$p\times q$ 複因子試驗之理論與分析

ON THE ANALYSIS OF P×Q FACTORIAL EXPERIMENT

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一、引　　　言

在田間試驗中，如同時參入兩個以上試因，每試因各俱若干變級，則可將不同試因之各變級彼此作完全之組合，此即組成複因子試驗之設計。此爲試驗之一重要項目。一般試驗設計之分析可自二種方式，其一使用一般之數理模型（見 Kempthorne 1955），另一則自逢機測驗模型 (Pitman 1937, Welch 1937)。複因子試驗由一般數理模型而論其分析者可見 Kempthorne (1956)。本文則自逢機測驗模型觀點，對於 $p\times q$ 之複因子試驗加以系統處理。此爲本文之目的。唯筆者學識淺薄，難免有未周到，尚請諸大方不吝指正。

二、$p\times q$ 複因子試驗之設計

設某一試因有 p 變級，另一試因有 q 變級，則完全組合共有 $p\times q$ 種組合；以此爲基礎再行逢機區集排列，即得 $p\times q$ 複因子試驗設計。此處小區總數共 $N=pqr$，r 爲重複次數，每區集內 pq 個小區。由逢機之原理，在 r 重複內每一組合佔據任何一小區之機會相等，共有 pq 種方式，故每一重複內 pq 個組合便有 $(pq)^2$ 種組合方式，r 區集共 $(pq)^2r$ 個組合。逢機測驗模型即根據此原則而構成。

三、數理模型之構成

爲構成本文分析之逢機測驗數理模型，設 y_{ijkl} 爲 (i,j) 組合於 k 區集內第 l 小區之實際收量，則 y_{ijkl} 可寫成下式：

$$\begin{aligned} y_{ijkl}=&y_{....}+(y_{ij..}-y_{....})+(y_{..kl}-y_{..k.})+(y_{..k.}-y_{....})\\ &+(y_{ijkl}-y_{..kl}-y_{ijk.}+y_{..k.})+(y_{ijk.}-y_{ij..}-y_{..k.}+y_{....}) \quad\cdots\cdots(1) \end{aligned}$$

上式中小寫字母之 y，及其下之點號表示平均數值，其各項之意義爲；

（1）$\mu=y_{....}$　爲全試驗各小區之總平均

（2）$\alpha_{ij}=y_{ij..}-y_{....}$　爲 (i,j) 組合就各小區之平均值與總平均值之偏差，表示 (i,j) 組合之效果。

（3）$p_{l(k)}=y_{..kl}-y_{..k.}$　爲第 k 區集內第 l 小區就各組合之平均值與第 k 區集內就各小區及組合之平均值之偏差，表示第 k 區集內第 l 小區地力之效果。

（4）$\gamma_k=y_{..k.}-y_{....}$　爲第 k 區集就各小區及組合之平均與總平均之偏差，表示第 k 區集地力之效果。

（5）$\eta_{ijl(k)}=y_{ijkl}-y_{..kl}-y_{ijk.}+y_{..k.}$　爲第 k 區集內 (i,j) 組合在第 l 小區之效果，用以表示 k 區集內 (i,j) 組合與第 l 小區之交互作用效果。

（6）$(\alpha\gamma)_{ijk}=l_{ijkl}=y_{ijk.}-y_{ij..}-y_{..k.}+y_{....}$　爲 (i,j) 組合於 k 區集內之效果，表示 (i,j) 組合與 k 區集之交互作用。

茲假定在試驗期間，試驗材料均爲均一化，此如試驗所用種子大小均一；試驗期間之耕

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作如播種，浸種，施肥等作業均一化；如此這些試因以外的原因可以略而不計，統歸之於機差。故在 $(pq)^2r$ 個組合中，可視 (ij,k,l) 組合爲 Euclidean 三度空間內之一點。在此玆導入一個機率變值 Y_{ijkl} (其實現値爲 y_{ijkl})。玆設 y_{ijkl} 變異於其平均 u_{ijkl} 之周圍，其變方爲 $\sigma^2{}_{ijkl}$。則在 (ij,k,l) 組合內，可得

$$y_{ijkl}=u_{ijkl}+e_{ijkl}.$$

$$\mathrm{E}\ (e_{ijkl})=0$$

$$\mathrm{E}\ (e^2{}_{ijkl})=\sigma^2{}_{ijkl}$$

故知 $$\frac{1}{(pq)^2r}\sum_i^p\sum_j^q\sum_k^r\sum_l^{pq}y_{ijkl}=y_{....}=u_{....}+e_{....}$$

$$\frac{1}{pqr}\sum_k^r\sum_l^{pq}y_{ijkl}=y_{ij..}=u_{ij..}+e_{ij..}$$

$$\frac{1}{pq}\sum_i^p\sum_j^q y_{ijkl}=y_{..kl}=u_{..kl}+e_{..kl}$$

$$\frac{1}{(pq)^2}\sum_i^p\sum_j^q\sum_l^{pq}y_{ijkl}=y_{..k.}=u_{..k.}+e_{..k.}$$

$$\frac{1}{pq}\sum_l^{pq}y_{ijkl}=y_{ijk.}=u_{ijk.}+e_{ijk.}$$

式中 $$u_{ijk.}=\frac{1}{pq}\sum_l^{pq}u_{ijkl}\qquad e_{ijk.}=\frac{1}{pq}\sum_l^{pq}e_{ijkl}$$

$$u_{ij..}=\frac{1}{r}\sum_k^r u_{ijk.}\qquad e_{ij..}=\frac{1}{r}\sum_k^r e_{ijk.}$$

$$u_{..k.}=\frac{1}{pq}\sum_i^p\sum_j^q u_{ijk.}\qquad e_{..k.}=\frac{1}{pq}\sum_i^p\sum_j^q e_{ijk.}$$

$$u_{....}=\frac{1}{r}\sum_k^r u_{..k.}\qquad e_{....}=\frac{1}{r}\sum_k^r e_{..k.}$$

玆令 $\alpha_{ij}=u_{ij..}-u_{....}$ 而以 $y_{ij..}-y_{....}$ 估算之

$r_k=u_{..k.}-u_{....}$ 以 $y_{..k.}-y_{....}$ 估算之

$p_{l(k)}=u_{..kl}-u_{..k.}$ 以 $y_{..kl}-y_{..k.}$ 估算之

$n_{ijl(k)}=u_{ijkl}-u_{..kl}-u_{ijk.}+u_{..k.}$ 而以 $y_{ijkl}-y_{..kl}-y_{ijk.}+y_{..k.}$ 估算之

在此試驗設計中，係以各組合與重複之互作估算機差。y_{ijkl} 之機差爲 e_{ijkl}，故 y_{ijkl} 可以下列模式表示之

$$y_{ijkl}=u_{....}+\alpha_{ij}+r_k+p_{l(k)}+n_{ijl(k)}+e_{ijkl}\cdots\cdots(2)$$

次再就 (i,j) 組合作進一步分析爲 A 試因之效果及 B 試因之效果，再加上 A 試因與 B 試因之互交效應。

玆將 $y_{ij..}-y_{....}$ 寫成

$$y_{ij..}-y_{....}=(y_{i...}-y_{....})+(y_{.j..}-y_{....})+(y_{ij..}-y_{i...}-y_{.j..}+y_{....})$$

並設 $$u_{i...}=\frac{1}{q}\sum_j^q u_{ij..}$$

$$u_{.j..}=\frac{1}{p}\sum_i^p u_{ij..}$$

$$u_{\cdots\cdot}=\frac{1}{pq}\sum_i^p\sum_j^q u_{ij\cdot\cdot}$$

故 A 試因第 i 水準之效應為

$a_i=\mu_{i\cdots}-u_{\cdots\cdot}$ 以 $y_{i\cdots}-y_{\cdots\cdot}$ 估算之

B 試因第 j 水準之效應為

$b_j=\mu_{\cdot j\cdot\cdot}-u_{\cdots\cdot}$ 以 $y_{\cdot j\cdot\cdot}-y_{\cdots\cdot}$ 估算之

A 試因第 i 水準與 B 試因第 j 水準之交互作用效果為

$(ab)_{ij}=\mu_{ij\cdot\cdot}-\mu_{i\cdots}-\mu_{\cdot j\cdot\cdot}+\mu_{\cdots\cdot}$ 而以 $(y_{ij\cdot\cdot}-y_{i\cdots}-y_{\cdot j\cdot\cdot}+y_{\cdots\cdot})$ 估算之

由此上式模式變成

$$\begin{aligned}y_{ijkl}&=\mu_{\cdots\cdot}+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\gamma_k+p_{l(k)}+n_{ijl(k)}+e_{ijkl}\\&=\mu_{\cdots\cdot}+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\gamma_k+z_{ijkl}(=p_{l(k)}+n_{ijl(k)}+e_{ijkl})\end{aligned}\quad\cdots\cdots\cdots(3)$$

此時顯然成立 $\sum_i^p\alpha_i=0,\quad \sum_j^q\beta_j=0,\quad \sum_i^p(\alpha\beta)_{ij}=\sum_j^q(\alpha\beta)_{ij}=0$

$$\sum_k^r\gamma_k=0,\quad \sum_l^{pq}p_{l(k)}=0,\quad \sum_{ij}^{pq}n_{ijl(k)}=\sum_l^{pq}n_{ijl(k)}=0$$

使用此模型而進行分析時，則所需要的前提為：

(1) 各種效應之間互有相加性。

(2) 各種效應之間互為獨立性，此點實際存在而可以證明之。

在顯著性測驗時，是否需要常態性；此點將於以後詳細說明。

(三)逢機之作用。觀以上數理模型(3)，逢機作用僅及於 $z_{ijkl}(=p_{l(k)}+n_{ijl(k)}+e_{ijkl})$ 其他各項效應則不受逢機化之影響。為說明逢機化之原理，玆特引入一個新機率變量 $\delta^l{}_{ijk}$, $\delta^l{}_{ijk}$ 之分布受逢機化所決定。

$\delta^l{}_{ijk}=1$, 如在 k 區集內 (i,j) 組合存在於第 l 小區

$\quad\quad=0$, 第 k 區集內 (i,j) 組合不在第 l 小區

引入此 $\delta^l{}_{ijk}$ 之後，令 y_{ijk} 為 (i,j) 組合在 k 區集內之實際收量，則顯然可見

$$\begin{aligned}y_{ijk}&=\mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\gamma_k+\sum_l\delta^l{}_{ijk}p_{l(k)}\\&\quad+\sum_l\delta^l{}_{ijk}\eta_{ijl(k)}+\sum_l\delta^l{}_{ijk}e_{ijkl}\end{aligned}\quad\cdots\cdots\cdots(4)$$

$\delta^l{}_{ijk}$ 則服從以下法則，

$$\left.\begin{array}{l}p_r.(\delta^l{}_{ijk}=1)=\dfrac{1}{pq},\ i=1,2\cdots p,\ j=1,2\cdots q,\ k=1,2\cdots r\\ \qquad\qquad\qquad\quad l=1,2\cdots,pq.\\ p_r.(\delta^l{}_{ijk}=1\mid\delta^{l'}{}_{ijk}=1)=0\qquad l\neq l'\\ p_r.(\delta^l{}_{ijk}=1\mid\delta^l{}_{i'j'k}=1)=0\qquad ij\neq i'j'\ \begin{cases}i=i',\ j\neq j'\\ i\neq i',\ j=j'\\ i\neq i',\ j\neq j'\end{cases}\\ p_r.(\delta^l{}_{ijk}=1\mid\delta^{l'}{}_{i'j'k'}=1)=\dfrac{1}{p^2q^2}\qquad k\neq k'\\ p_r.(\delta^l{}_{ijk}=1\mid\delta^{l'}{}_{i'j'k}=1)=\dfrac{1}{pq(pq-1)}\quad l\neq l',\quad ij\neq i'j'\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\begin{cases}i\neq i',\ j=j'\\ i=i',\ j\neq j'\\ i\neq i',\ j\neq j'\end{cases}\end{array}\right\}\cdots\cdots\cdots(5)$$

以上各機率係由於各區集內各組合之落在小區內可能位置的變動所造成。上列各機率顯然指出幾點原則，即 (a) 在同一區集內同一組合不能同時佔據兩個以上小區，(b) 在同一區

集內不同組合亦不能同時佔據同一小區。(*c*) 同一區集內不同組合佔據不同小區則依據順序排列之原則 (Permutation)。(*d*) 對於不同區集，則各組合佔據各區之事象互爲獨立。由此 $\delta^{l}{}_{ijk}$ 之期望値爲

$$
\left.\begin{aligned}
&E(\delta^{l}{}_{ijk})=\frac{1}{pq},\ E(\delta^{l2}{}_{ijk})=\frac{1}{pq}\ \rightarrow\ E(\delta^{l}{}_{ijk}{}^{r})=\frac{1}{pq},\ r=\text{正整數}\\
&E(\delta^{l}{}_{ijk}\,\delta^{l}{}_{i'j'k})=0,\rightarrow E[(\delta^{l}{}_{ijk})^{p'}(\delta^{l}{}_{i'j'k})^{q'}]=0,\ ij\neq i'j',\ p',q'\ \text{正整數}\\
&E(\delta^{l}{}_{ijk}\,\delta^{l'}{}_{ijk})=0,\ \rightarrow\ E[(\delta^{l}{}_{ijk})^{p'}(\delta^{l'}{}_{ijk})^{q'}]=0,\ l\neq l',\ p',q'\ \text{正整數}\\
&E(\delta^{l}{}_{ijk}\,\delta^{l'}{}_{i'j'k})=\frac{1}{pq(pq-1)},\rightarrow E[(\delta^{l}{}_{ijk})^{p'}(\delta^{l'}{}_{i'j'k})^{q'}]\\
&\qquad\qquad=\frac{1}{pq(pq-1)},\quad ij\neq i'j',\quad i\neq i',\quad p',q'\ \text{正整數}\\
&E[(\delta^{l}{}_{ijk})^{p_1}(\delta^{l^1}{}_{i'j'k})^{p_2}\cdots(\delta_{l^{m-1}}{}_{i^{m-1}}{}_{j^{m-1}}{}_{k})^{p_m}]\\
&\quad=\frac{1}{pq(pq-1)\cdots(pq-m+1)},\quad p_1,p_2,\cdots p_m\ \text{正整數}\\
&E[\delta^{l}{}_{ijk}\,\delta^{l'}{}_{i'j'k'}]=\frac{1}{p^2q^2},\rightarrow k\neq k'\,E[(\delta^{l'}{}_{ijk})^{p'}(\delta^{l}{}_{i'j'k'})^{q'}]=\frac{1}{p^2q^2},\\
&\ k\neq k'\rightarrow E[(\delta^{l}{}_{ijk})^{p_1}(\delta^{l'}{}_{i'j'k'})^{p_2}\cdots(\delta_{l^{m-1}}{}_{i^{m-1}}{}_{j^{m-1}}{}_{k^{m-1}})^{p_m}]\\
&\quad=\frac{1}{p^mq^m},\quad k\neq k'\neq\cdots\neq k^{m-1},
\end{aligned}\right\}\cdots\cdots(6)
$$

四、動 差 之 計 算

茲令 $\varepsilon_{ijk}=\sum\limits_{l}\delta^{l}{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})$，則對於 ε_{ijk} 其各級之動差可計算之如下示：

$$
\begin{aligned}
E(\varepsilon_{ijk})&=\underset{l}{E}(\sum\delta^{l}{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})]\\
&=\frac{1}{pq}\sum_{l}E(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})=0\cdots\cdots\cdots\cdots(7)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(\varepsilon_{ijk})&=E(\varepsilon^{2}{}_{ijk})=E[\sum_{l}\delta^{l2}{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})^2\\
&\quad+\sum_{l\neq l'}\sum\delta^{l}{}_{ijk}\delta^{l'}{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})(p_{l'(k)}+\eta_{ijl'(k)}+e_{ijkl'})]\\
&=\frac{1}{pq}\sum_{l}[(p^{2}{}_{l(k)}+\eta^{2}{}_{ijl(k)})+\sigma^{2}{}_{ijkl}]\cdots\cdots\cdots\cdots(8)
\end{aligned}
$$

$$
\mathrm{Cov}(\varepsilon_{ijk}\varepsilon_{i'jk})=\frac{-1}{pq(pq-1)}[\sum_{l}p^{2}{}_{l(k)}+\sum_{l}\eta_{ijl(k)}\eta_{i'j'l(k)}]\cdots\cdots\cdots\cdots(10)
$$

$$
\begin{cases}i'\neq i,\ j\neq j'\\ i=i',\ j\neq j'\\ \text{或}\ i\neq i'\ \ j=j'\end{cases}
$$

$$
\mathrm{Cov}(\varepsilon_{ijk}\varepsilon_{i'j'k'})=0,\ k\neq k'\ \cdots\cdots\cdots\cdots(11)
$$

又

$$
\begin{aligned}
&E(\varepsilon_{i\cdot k})=E(\sum_{j}\sum_{l}\delta^{l}{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})]=0.\ \ E(\varepsilon_{i\cdot\cdot})=E(\varepsilon_{\cdot j\cdot})=0\\
&\quad E(\varepsilon_{\cdot jk})=0,\ \ E(\varepsilon_{\cdot\cdot k})=0,\ \ E(\varepsilon_{\cdot\cdot\cdot})=0,\ \ E(\varepsilon_{ij\cdot})=0\ \cdots\cdots\cdots\cdots(12)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(\varepsilon_{i\cdot k})&=E(\varepsilon^{2}{}_{i\cdot k})=E[(\sum_{j}\sum_{l}\delta^{l}{}_{ijk}(p_{l(k)}\eta_{ijl(k)}+e_{ijkl}))^2]\\
&=E[\sum_{j\neq j'}\sum\ \sum_{l\neq l'}\sum\delta^{l}{}_{ijk}\delta^{l'}{}_{ij'k}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})(p_{l'(k)}+\eta_{ij'l'(k)}+e_{ij'kl'})\\
&\quad+\sum_{j}\sum_{l}\delta^{l2}{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})^2]\\
&=\frac{1}{pq(pq-1)}\sum_{j\neq j'}\sum\ \sum_{l}(p_{l(k)}+\eta_{ijl(k)})(-p_{l(k)}-\eta_{ij'l(k)})
\end{aligned}
$$

$$+\frac{1}{pq}\sum_j\sum_l[(p_{l(k)}+\eta_{ijl(k)})^2+\sigma^2{}_{ijkl}]$$

$$=\frac{-1}{pq(pq-1)}(\sum_j\sum_l(p_{l(k)}+\eta_{ijl(k)})[(q-1)p_{l(k)}+q\eta_{i\cdot l(k)}-\eta_{ijl(k)}]$$

$$+\frac{1}{pq}\sum_j\sum_l[p^2{}_{l(k)}+\eta^2{}_{ijl(k)}+\sigma_{ijkl}]$$

$$=\frac{-1}{pq(pq-1)}\sum_j\sum_l[(q-1)p^2{}_{l(k)}-\eta^2{}_{ijl(k)}]-\frac{1}{pq(pq-1)}q^2\sum_l\eta^2{}_{i\cdot l(k)}$$

$$+\frac{1}{pq}\sum_j\sum_l[p^2{}_{l(k)}+\eta^2{}_{ijl(k)}+\sigma^2{}_{ijkl}]$$

$$=\frac{p-1}{p(pq-1)}\sum_l p^2{}_{l(k)}+\frac{1}{(pq-1)}\sum_j\sum_l\eta^2{}_{ijl(k)}+\frac{1}{pq}\sum_j\sum_l\sigma^2{}_{ijkl}$$

$$-\frac{q}{p(pq-1)}\sum_l\eta^2{}_{i\cdot l(k)}\quad\cdots\cdots(13)$$

同理得

$$\text{Var. }(\varepsilon_{\cdot jk})=\frac{(q-1)}{q(pq-1)}\sum_l p^2{}_{l(k)}+\frac{1}{(pq-1)}\sum_i\sum_l\eta^2{}_{ijl(k)}$$

$$+\frac{1}{pq}\sum_i\sum_l\sigma^2{}_{ijkl}-\frac{p}{q(pq-1)}\sum_l\eta^2{}_{\cdot jl(k)}\quad\cdots\cdots(14)$$

而 $\text{Var. }(\varepsilon_{\cdot\cdot k})=E[(\sum_i\sum_j\sum_l\delta^l{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl}))^2]$

$$=\frac{1}{pq-1}\sum_i\sum_j\sum_l\eta^2{}_{ijl(k)}+\frac{1}{pq}\sum_i\sum_j\sum_l\sigma^2{}_{ijkl},\quad(\text{因 }\sum_l\eta_{ijl(k)}=0)$$

$$\text{Var }(\varepsilon_{i\cdots})=E[(\sum_j\sum_k\sum_l\delta^l{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl}))^2]$$

$$=\frac{1}{pq}\sum_j\sum_k\sum_l\sigma^2{}_{ijkl}+\frac{rq^2(p-1)}{pq(pq-1)}\sum_l p^2{}_{l(k)}+\frac{1}{pq-1}\sum_k\sum_j\sum_l\eta^2{}_{ijl(k)}$$

$$-\frac{q}{p(pq-1)}\sum_k\sum_l\eta^2{}_{i\cdot l(k)}$$

$$\text{Var }(\varepsilon_{\cdot j\cdot})=\frac{1}{pq}\sum_i\sum_k\sum_l\sigma^2{}_{ijkl}+\frac{rp^2(q-1)}{pq(pq-1)}\sum_l p^2{}_{l(k)}+\frac{1}{pq-1}\sum_i\sum_k\sum_l\eta^2{}_{ijl(k)}$$

$$-\frac{p}{q(pq-1)}\sum_k\sum_l\eta^2{}_{\cdot jl(k)}$$

$$\text{Var. }(\varepsilon_{ij\cdot})=E[\sum_k\sum_l\delta^l{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})]^2$$

$$=\frac{1}{pq}\sum_k\sum_l[p^2{}_{l(k)}+\eta^2{}_{ijl(k)}+\sigma^2{}_{ijkl}]$$

$$\text{Var. }(\varepsilon_{\cdots})=E[\sum_i\sum_j\sum_k\sum_l\delta^l{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e^l{}_{ijk})]^2$$

$$=\frac{1}{pq-1}\sum_i\sum_j\sum_k\sum_l\eta^2{}_{ijl(k)}+\frac{1}{pq}\sum_i\sum_j\sum_k\sum_l\sigma^2{}_{ijkl}$$

$$\underset{k\neq k'}{\text{Cov}}(\varepsilon_{i\cdot k}\varepsilon_{i'\cdot k'})=E(\varepsilon_{i\cdot k}\varepsilon_{i'\cdot k'})=E(\varepsilon_{i\cdot k})E({}_{i'\cdot k'})=0\quad\cdots\cdots(15)$$

$$\underset{k\neq k'}{\text{Cov}}(\varepsilon_{\cdot jk}\varepsilon_{\cdot j'k'})=\underset{k\neq k'}{\text{Cov}}(\varepsilon_{\cdot\cdot k}\varepsilon_{\cdot\cdot k'})=0$$

$$\text{Cov }(\varepsilon_{i'\cdot k}\varepsilon_{i\cdot k})=E(\varepsilon_{i'\cdot k}\varepsilon_{i\cdot k})\quad(i\neq i')$$

$$=E[\sum_j\sum_l\delta^l{}_{ijk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})]\times[\sum_j\sum_l\delta^l{}_{i'jk}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})]$$

$$=\sum_j\sum_{j'}\sum_{l\neq l'}\sum\delta^l{}_{ijk}\delta^{l'}{}_{i'j'k}(p_{l(k)}+\eta_{ijl(k)}+e_{ijkl})(p_{l'(k)}+\eta_{i'j'l'(k)}+e_{i'j'kl'})$$

$$= \frac{1}{pq(pq-1)} \sum_{j} \sum_{j'} \sum_{l} (p_{l(k)} + \eta_{ijl(k)}) \times (-p_{l(k)} - \eta_{i'j'l(k)})$$

$$= \frac{-1}{pq(pq-1)} [q^2 \sum_{l} p^2{}_{l(k)} + q^2 \sum_{l} \eta_{i \cdot l(k)} \eta_{i' \cdot l(k)}] \cdots\cdots\cdots\cdots\cdots\cdots (16)$$

同理 $C_{ov}(\underset{j \neq jl}{\varepsilon_{\cdot jk} \cdot \varepsilon_{\cdot j'k}}) = \frac{-1}{pq(pq-1)} [p^2 \sum_{l} p^2{}_{l(k)} + p^2 \sum_{l} \eta_{\cdot jl(k)} \eta_{\cdot j'l(k)}]$

五、各平方和之期望值與變方分析

(3) 式就所有個體數相加則得

$$Y_{\cdots} = \sum_{i} \sum_{j} \sum_{k} Y_{ijk} = pqr\mu + \sum_{i} \sum_{j} \sum_{k} (\sum_{l} \delta^{l}{}_{ijk} (p_{l(k)} + \eta_{ijl(k)} + e_{ijkl}))$$

因 $E(\delta^{l}{}_{ijk}) = \frac{1}{pq}$, 而 $\sum p_{l(k)} = 0$, $\sum \eta_{ijl(k)} = 0$, $\sum e_{ijkl} = 0$.

故 $E(\frac{Y_{\cdots}}{pqr} = \hat{\mu}) = \mu$.

同理就所有 A 試因第 i 變級者相加則得

$$Y_{i\cdot\cdot} = qr\mu + qra_i + \sum_{j} \sum_{k} (\sum_{l} \delta^{l}{}_{ijk} (p_{l(k)} + \eta_{ijl(k)} + e_{ijkl}))$$

故得 $\hat{\alpha}_i = \frac{Y_{i\cdot\cdot}}{qr} - \frac{Y_{\cdots}}{pqr}$, 而 $E(\hat{\alpha}_i) = \alpha_i$

同理 $Y_{\cdot j \cdot} = pr\mu + prb_j + \sum_{i} \sum_{k} (\sum_{l} \delta^{l}{}_{ijk} (p_{l(k)} + \eta_{ijl(k)} + e_{ijkl}))$ ……(17)

故 $\hat{\beta}_j = \frac{Y_{\cdot j \cdot}}{pr} - \frac{Y_{\cdots}}{pqr}$ 而 $E(\hat{\beta}_j) = \beta_j$

$$Y_{ij\cdot} = r\mu + ra_i + rb_j + r(ab)_{ij} + \sum_{k} (\sum_{l} \delta^{l}{}_{ijk} (p_{l(k)} + \eta_{ijl(k)} + e_{ijkl})$$

故 $(\alpha\beta)_{ij} = \frac{Y_{ij\cdot}}{r} - \frac{Y_{i\cdot\cdot}}{qr} - \frac{Y_{\cdot j \cdot}}{pr} + \frac{Y_{\cdots}}{pqr}$ 而 $E(\widehat{\alpha\beta})_{ij} = (\alpha\beta)_{ij}$

$$Y_{\cdot\cdot k} = pq\mu + pqr_k + \sum_{i} \sum_{j} (\sum_{l} \delta^{l}{}_{ijk} (p_{l(k)} + \eta_{ijl(k)} + e_{ijkl}))$$

故 $\hat{r}_k = \frac{Y_{\cdot\cdot k}}{pq} - \frac{Y_{\cdots}}{pqr}$, $E(\hat{r}_k) = r_k$

各效應之平方和爲

（1） 由於總平均之部份

$$Y_{\cdots} \times \hat{\mu} = \frac{Y_{\cdots}{}^2}{pqr}$$

（2） 由於 A 試因之部份

$$\sum_{i} \hat{a} \times Y_{i\cdot\cdot} = \sum_{i} Y_{i\cdot\cdot} \left(\frac{Y_{i\cdot\cdot}}{qr} - \frac{Y_{\cdots}}{pqr} \right) = \sum_{i} \frac{Y_{i\cdot\cdot}{}^2}{pr} - \frac{Y_{\cdots}{}^2}{pqr}$$

（3） 由於B試因之部份

$$\sum_{j} \hat{b}_j Y_{\cdot j \cdot} = \sum_{j} Y_{\cdot j \cdot} \left(\frac{Y_{\cdot j \cdot}}{pr} - \frac{Y_{\cdots}}{pqr} \right) = \sum_{j} \frac{Y^2{}_{\cdot j \cdot}}{pr} - \frac{Y^2{}_{\cdots}}{pqr}$$

（4） 由於A試因與B試因互作之部份

$$\sum_{i} \sum_{j} (\widehat{ab})_{ij} Y_{ij\cdot} = \sum_{i} \sum_{j} Y_{ij\cdot} \left(\frac{Y_{ij\cdot}}{r} - \frac{Y_{i\cdot\cdot}}{qr} - \frac{Y_{\cdot j \cdot}}{pr} + \frac{Y_{\cdots}}{pqr} \right)$$

$$= \left(\sum_{i} \sum_{j} \frac{Y^2{}_{ij\cdot}}{r} - \frac{Y^2{}_{\cdots}}{pqr} \right) - \left(\sum_{i} \frac{Y^2{}_{i\cdot\cdot}}{qr} - \frac{Y^2{}_{\cdots}}{pqr} \right)$$

$$- \left(\sum_{j} \frac{Y^2{}_{\cdot j \cdot}}{pr} - \frac{Y^2{}_{\cdots}}{pqr} \right)$$

(5) 由於區集土異之部份

$$\sum_k \hat{r}_k Y_{..k} = \sum_k Y_{..k}\left(\frac{Y_{..k}}{pq} - \frac{Y_{...}}{pqr}\right) = \sum_k \frac{Y^2_{..k}}{pq} - \frac{Y^2_{...}}{pqr}$$

其各項平方和之期望值

(1) 由於總平均之部份

$$E\left(\frac{Y^2_{...}}{pqr}\right) = \frac{1}{pqr} E[pqr\mu + \sum_i\sum_j\sum_k\sum_l \delta^l{}_{ijk}(p_{l(k)} + \eta_{ijl(k)} + e_{ijkl})]^2$$

$$= pqr\mu^2 + \frac{1}{pqr} V_{ar.}(\varepsilon_{...}) = pqr\mu^2 + \frac{1}{(pq)^2 r} \sum_i\sum_j\sum_k\sum_l \sigma^2{}_{ijkl}$$

$$+ \frac{1}{pqr(pq-1)} \sum_i\sum_j\sum_k\sum_l \eta^2{}_{ijl(k)} \quad \cdots\cdots(18)$$

又因

$$E\left[\sum_i \frac{Y^2_{i..}}{qr}\right] = \frac{1}{qr}\sum_i E[qr\mu + qr\alpha_i + \sum_j\sum_k\sum_l \delta^l{}_{ijk}(p_{l(k)} + \eta_{ijl(k)} + e_{ijkl})]^2$$

$$= pqr\mu^2 + qr\sum_i \alpha_i^2 + \frac{1}{qr}\sum_i \mathrm{Var}(\varepsilon_{i..})$$

$$= pqr\mu^2 + qr\sum_i \alpha_i^2 + p\frac{1}{(pq)^2 r}\sum_i\sum_j\sum_k\sum_l \sigma^2{}_{ijkl} + \frac{1}{qr(pq-1)}\sum_i\sum_j\sum_k\sum_l \eta^2{}_{ijl(k)}$$

$$+ \frac{(p-1)}{(pq-1)}\sum_l p^2{}_{l(k)} - \frac{q}{pqr(pq-1)}\sum_i\sum_k\sum_l \eta^2{}_{i\cdot l(k)} \quad \cdots\cdots(19)$$

(2) 故知A試因效果平方和之期望值爲

$$E\left(\sum_i \frac{Y^2_{i..}}{qr}\right) - E\left(\frac{Y^2_{...}}{pqr}\right) = qr\sum_i \alpha_i^2 + (p-1)\frac{1}{(pq)^2 r}\sum_i\sum_j\sum_k\sum_l \sigma^2{}_{ijkl}$$

$$+ \frac{(p-1)}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l \eta^2{}_{ijl(k)} + \frac{p-1}{(pq-1)}\sum_l p^2{}_{l(k)}$$

$$- \frac{q}{pqr(pq-1)}\sum_i\sum_k\sum_l \eta^2{}_{i\cdot l(k)} \quad \cdots\cdots(20)$$

(3) 同理得B試因效果平方和之期望值爲

$$E\left(\sum_j \frac{Y^2_{.j.}}{pr}\right) - E\left(\frac{Y^2_{...}}{pqr}\right) = pr\sum_j \beta_j^2 + (q-1)\frac{1}{(pq)^2 r}\sum_i\sum_j\sum_k\sum_l \sigma^2{}_{ijkl}$$

$$+ \frac{(q-1)}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l \eta^2{}_{ijl(k)} + \frac{q-1}{pq-1}\sum_l p^2{}_{l(k)}$$

$$- \frac{p}{pqr(pq-1)}\sum_j\sum_k\sum_l \eta^2{}_{\cdot jl(k)} \quad \cdots\cdots(21)$$

(4) A×B 互作平方和之期望值爲

$$E\left[\sum_i\sum_j \frac{Y^2_{ij.}}{r}\right] - E\left[\frac{Y^2_{...}}{pqr}\right] - E(SSA) - E(SSB)$$

$$= r\sum_i\sum_j (\alpha b)^2{}_{ij} + (p-1)(q-1)\frac{1}{(pq)^2 r}\sum_i\sum_j\sum_k\sum_l \sigma^2{}_{ijkl}$$

$$+ \frac{(p-1)(q-1)}{pq-1}\sum_l p^2{}_{l(k)} + \frac{(p-1)(q-1)}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l \eta^2{}_{ijl(k)}$$

$$- \frac{1}{pq-1}\sum_i\sum_j\sum_k\sum_l (\eta_{ijl(k)} - \eta_{i\cdot l(k)} - \eta_{\cdot jl(k)})^2 \quad \cdots\cdots(22)$$

區集平方和之期望值爲

$$pq\sum_k r^2_k+(r-1)\frac{1}{(pq)^2r}\sum_i\sum_j\sum_k\sum_l\sigma^2_{ijkl}$$
$$+\frac{(r-1)}{pq-1}\sum_l p^2_{l(k)}+\frac{r-1}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l\eta^2_{ijl(k)}\quad\cdots\cdots(23)$$

機差平方和之期望值則爲

$$\frac{(pq-1)(r-1)}{(pq)^2r}\sum_i\sum_j\sum_k\sum_l\sigma^2_{ijkl}+\frac{(pq-1)(r-1)}{pq-1}\sum_l p^2_{l(k)}$$
$$+\left(\frac{pq-2}{pq-1}\right)\frac{(pq-1)(r-1)}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l\eta^2_{ijl(k)}\quad\cdots\cdots(24)$$

如令 $\sigma_e{}^2=\frac{1}{(pq)^2r}\sum_i\sum_j\sum_k\sum_l\sigma^2_{ijkl}$

$\sigma_r{}^2=\frac{1}{r-1}\sum_k r_k{}^2$ 　則得下列變方分析表

$p\times q$ 複因子試驗變方分析表

Analysis of Variance Table in $p\times q$ Factorial Experiment

(Under Randomization Model)

變因 Sources of Variation	自由度 Degrees of Freedom $=d.f$	平方和 Sum of Squares=S. S.	均方 Mean Square =M. S.	均方期望值 Expectation of Mean Square
區集(Block)	$r-1$	$\mathrm{SSR}=\sum_k^r\frac{Y^2_{\cdot\cdot k}}{pq}-\frac{Y^2_{\cdots}}{pqr}$	$\mathrm{R}=\mathrm{SSR}/r-1$	$pq\sigma_r{}^2+\sigma_e{}^2+\frac{1}{(pq-1)}\sum_l p^2_{l(k)}+\frac{1}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l(\eta_{ijl(k)})^2$
A試因	$p-1$	$\mathrm{SSA}=\sum_i^p\frac{Y^2_{i\cdot\cdot}}{qr}-\frac{Y^2_{\cdots}}{pqr}$	$\mathrm{A}=\mathrm{SSA}/p-1$	$qr\left(\frac{1}{p-1}\sum_i\alpha_i{}^2\right)+\sigma_e{}^2+\frac{1}{pq-1}\sum_l p^2_{l(k)}+\frac{1}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l(\eta_{ijl(k)})^2-\frac{q}{(p-1)pqr(pq-1)}\sum_i\sum_j\sum_l(\eta_{i\cdot l(k)})^2$
B試因	$q-1$	$\mathrm{SSB}=\sum_j^q\frac{Y^2_{\cdot j\cdot}}{pr}-\frac{Y^2_{\cdots}}{pqr}$	$\mathrm{B}=\mathrm{SSB}/q-1$	$pr\left(\frac{1}{q-1}\sum_j\beta_j{}^2\right)+\sigma_e{}^2+\frac{1}{pq-1}\sum_l p^2_{l(k)}+\frac{1}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l(\eta_{ijl(k)})^2-\frac{p}{(q-1)pqr(pq-1)}\sum_j\sum_k\sum_l\eta^2_{\cdot jl(k)}$
A×B	$(p-1)(q-1)$	$\mathrm{SSA\times B}=\sum_i\sum_j\frac{Y^2_{ij\cdot}}{r}-\frac{Y^2_{\cdots}}{pqr}-\mathrm{SSA}-\mathrm{SSB}$	$\mathrm{A\times B}=\frac{\mathrm{SSA\times B}}{(p-1)(q-1)}$	$r\left(\frac{1}{(p-1)(q-1)}\sum_i\sum_j(\alpha\beta)^2_{ij}\right)+\sigma_e{}^2+\frac{1}{pq-1}\sum_l p^2_{l(k)}+\frac{1}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l\eta^2_{ijl(k)}-\frac{1}{(p-1)(q-1)pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l(\eta_{ijl(k)}-\eta_{i\cdot l(k)}-\eta_{\cdot jl(k)})^2$

機差	$(pq-1)(r-1)$	$SSE=\sum_j\sum_i\sum_k Y^2_{ijk}-\frac{Y^2_{\cdots}}{pqr}-SSR-SSA-SSB-SSA\times B.$	$E=SSE/(pq-1)(r-1)$	$\sigma_e^2+\frac{1}{pq-1}\sum_l p^2_{l(k)}+\left(\frac{pq-2}{pq-1}\right)\frac{1}{pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l \eta^2_{ijl(k)}$

六、關於 ε_{ijk} 之假定及顯著性測定

進行顯著性測定時對 ε_{ijk} 可有三種不同假定方式，此爲 Hatamura, Okuno 與 Sasaki (1954) 在逢機區集設計方析上所討論者。此三假定前提依次爲：

(a) ε_{ijk} 之一次，二次，三次，四次動差均存在，但對於 ε_{ijk} 不付與任何限制。ε_{ijk} 可能不是常態之機值。

(b) ε_{ijk} 服從多次元相關常態分布，$\eta_{ijl(k)}=0$, 各小區之機差變方亦相等，即 $\sigma_{ijkl}=\sigma$

(c) $\varepsilon_{ijk}\ \varepsilon$ NID $(0,\ \sigma^2)$, $\eta_{ijl(k)}=0$

對於以上三假定，(b), (c) 可歸爲一類。在本試驗設計上，使用最大可能比法 (Likelihood ratio Criteria) 可以證明實用於 (c) 假定之 F 測驗亦同樣能實行於 (b) 之前提。故在本文中 (b) 之假定不再討論。在 (a) 假定下，可使用 Pitman (1937) 氏方法以求得近似服從 F 分布時之條件。此種方法 Kempthorne (1952), Hatamura et al (1954) 曾應用於逢機區集之設計。

七、常態性之假定與顯著性測定

在假定 (c) 之前提下，本設計之數理模型可簡寫成

$$y_{ijk}=\mu+a_i+b_j+(ab)_{ij}+r_k+e_{ijk} \cdots\cdots(25)$$

y_{ijk} 係服從 NID $(\mu+a_i+b_j+(ab)_{ij}+r_k,\ \sigma^2)$

故最大可能函數爲

$$L=\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{\frac{N}{2}} e^{-\frac{1}{2\sigma^2}\sum_i\sum_j\sum_k(y_{ijk}-\mu-a_i-b_j-(ab)_{ij}-r_k)^2}$$

式中 $N=pqr$ 故

$$\log L=C-N\log\sigma^2-\frac{1}{2\sigma^2}\sum_i\sum_j\sum_k(y_{ijk}-\mu-a_i-b_j-(ab)_{ij}-r_k)^2$$

令 $\log L$ 爲最大，可得各估值爲（見前）

$$\hat{\mu}=\frac{Y_{\cdots}}{pqr}$$

$$\hat{a}_i=\frac{Y_{i\cdot\cdot}}{qr}-\frac{Y_{\cdots}}{pqr}$$

$$\hat{b}_j=\frac{Y_{\cdot j\cdot}}{pq}-\frac{Y_{\cdots}}{pqr}$$

$$(ab)_{ij}=\frac{Y_{ij\cdot}}{r}-\frac{Y_{i\cdot\cdot}}{qr}-\frac{Y_{\cdot j\cdot}}{pr}+\frac{Y_{\cdots}}{pqr}$$

$$\hat{r}_k=\frac{Y_{\cdot\cdot k}}{pq}-\frac{Y_{\cdots}}{pqr}$$

又 $$\frac{\partial\log L}{\partial\sigma^2}=-\frac{N}{\sigma^2}+\frac{1}{\sigma^4}\sum_i\sum_j\sum_k\left(Y_{ijk}-\frac{Y_{ij\cdot}}{r}-\frac{Y_{\cdot\cdot k}}{pq}+\frac{Y_{\cdots}}{pqr}\right)^2=0$$

故 $$\hat{\sigma}^2=\frac{1}{N}\sum_i\sum_j\sum_k\left(Y_{ijk}-\frac{Y_{ij\cdot}}{r}-\frac{Y_{\cdot\cdot k}}{pq}+\frac{Y_{\cdots}}{pqr}\right)^2$$

在 $H_0; (\alpha b)_{ij}=0 \quad i=1,2,3,\cdots p, j=1,2,3,\cdots q$ 之擬說下，

則 $\log L = C - N\log\sigma^2 - \frac{1}{2\sigma^2}\sum_i\sum_j\sum_k (Y_{ijk}-\mu-a_i-b_j-r_k)^2$

故 $\frac{\partial \log L}{\partial \mu}=0 \rightarrow \sum_i\sum_j\sum_k (Y_{ijk}-\mu-a_i-b_j-r_k)=0$

即 $Y_{\cdots}=pqr\hat{\mu}$

$$\frac{\partial \log L}{\partial \alpha_i}=0 \rightarrow Y_{i\cdot\cdot}=qr\hat{\mu}+qr\hat{\alpha}_i$$

$$\frac{\partial \log L}{\partial b_j}=0 \rightarrow Y_{\cdot j\cdot}=pr\hat{\mu}+pr\hat{b}_j$$

$$\frac{\partial \log L}{\partial \beta_k}=0 \rightarrow Y_{\cdot\cdot k}=pq\hat{\mu}+pq\hat{r}_k$$

故同樣得 $\hat{\mu}=\frac{Y_{\cdots}}{pqr}, \quad \hat{\alpha}_i=\frac{Y_{i\cdot\cdot}}{qr}-\frac{Y_{\cdots}}{pqr}$

$$\hat{b}_j=\frac{Y_{\cdot j\cdot}}{pr}-\frac{Y_{\cdots}}{pqr}, \quad \hat{r}_k=\frac{Y_{\cdot\cdot k}}{pq}-\frac{Y_{\cdots}}{pqr}$$

而 $\frac{\partial \log L}{\partial \sigma^2}=-\frac{N}{\sigma^2}+\frac{1}{\sigma^4}\sum_i\sum_j\sum_k\left(Y_{ijk}-\frac{Y_{i\cdot\cdot}}{qr}-\frac{Y_{\cdot j\cdot}}{pr}-\frac{Y_{\cdot\cdot k}}{pq}+\frac{2Y_{\cdots}}{pqr}\right)^2=0$

而 $\hat{\sigma}^2=\frac{1}{N}\sum_i\sum_j\sum_k\left(Y_{ijk}-\frac{Y_{i\cdot\cdot}}{qr}-\frac{Y_{\cdot j\cdot}}{pr}-\frac{Y_{\cdot\cdot k}}{pq}+\frac{2Y_{\cdots}}{pqr}\right)^2$

故得最大可能比 (Likelihood ratio Criteria) 爲

$$\lambda=\frac{P(\hat{w})}{P(\hat{\Omega})}=\left[\frac{\sum_i\sum_j\sum_k\left(Y_{ijk}-\frac{Y_{i\cdot\cdot}}{qr}-\frac{Y_{\cdot j\cdot}}{pr}-\frac{Y_{\cdot\cdot k}}{pq}+\frac{2Y_{\cdots}}{pqr}\right)^2}{\sum_i\sum_j\sum_k\left(Y_{ijk}-\frac{Y_{ij\cdot}}{r}-\frac{Y_{\cdot\cdot k}}{pq}+\frac{Y_{\cdots}}{pqr}\right)^2}\right]^{-\frac{N}{2}}$$

$$=\left(1+\frac{SSA\times B}{SSE}\right)^{-\frac{N}{2}}$$

$$=\left\{1+\frac{(p-1)(q-1)}{(pq-1)(r-1)}F[(p-1)(q-1),(pq-1)(r-1)]\right\}^{-\frac{N}{2}}$$

此處 $F=\frac{SSA\times B/(p-1)(q-1)}{SSE/(pq-1)(r-1)}$

此因在 $(\alpha b)_{ij}=0$ 之假定下，可以證出

SSA×B 服從 $x^2_{(p-1)(q-1)}\sigma^2$ 之分布，而

SSE 服從 $x^2_{(pq-1)(r-1)}\sigma^2$ 之分布，且 $x^2_{(p-1)(q-1)}$ 與 $x^2_{(pq-1)(r-1)}$ 互爲獨立 (證明省略)。故知欲檢定 $H_0, (\alpha b)_{ij}=0$ 之擬說，可利用自由度爲 $(p-1)(q-1), (pq-1)(r-1)$ 之 F 分布。其 Type I error (Level of Significance) 爲 0.05，危險域 (Critical Region) 爲 $F_0 \leqq F$ 如實得之 F 落於危險域 (Critical Region) 之內則拒絕 H_0，即表示 H_0; $(\alpha b)_{ij}=0$ 之擬說不能成立。反之承認 H_0。

同理測驗 $H_1, \alpha_i=0, i=1,2\cdots p.$

及 $H_2, \beta_j=0, j=1,2\cdots q$

時，同理可知可以利用

$F_1=\frac{SSA/p-1}{SSE/(pq-1)(r-1)}$ 以測驗 H_1, 自由度 $\begin{matrix} f_1=p-1 \\ f_2=(pq-1)(r-1) \end{matrix}$

$$F_2=\frac{SSB/q-1}{SSE/(pq-1)(r-1)}$$ 以測驗 H_2, $f_2=q-1$, $f_2=(pq-1)(r-1)$

欲求 Type II error 及此測驗之效能函數 (Power Function)，則可利用 Tang (1938) 之非心 F 分布 (Non-central F 以下以 G 表之)，其頻度函數爲

$$f(G)=\sum_{m=0}^{\infty}\frac{e^{-\lambda}\lambda^m}{m!\,B\left(\frac{f_1}{2}+m,\frac{\delta_2}{2}\right)}\left(\frac{G}{1+G}\right)^{\frac{f_1}{2}+m-1}\left(\frac{1}{1+G}\right)^{\frac{f_2}{2}+1}$$

在 H_0 下自由度爲 $f_1=(p-1)(q-1)$, $f_2=(pq-1)(r-1)$。B 表示 Beta 函數。而

$$\lambda=\frac{\sum_i\sum_j(\alpha b)^2{}_{ij}}{2\sigma^2}$$

如設立 H_0; $(\alpha b)_{ij}=0$, $i,=1,2\cdots p_1\ j=1,2\cdots q$，則 $\lambda=0$，非心 F 分布便成爲一般之 F 分布，自由度爲 f_1, f_2。設 $\lambda=a\neq0$，則爲非心 F 分布。故在 $\lambda=a\neq0$ 下 Type II error 爲

$$\beta=\int_0^{G_0=F_0}f(G\mid\lambda=a)dG$$

其效能函數則爲

$$p(\beta)=\int_{F_0}^{\infty}f(G\mid\lambda=a)dG$$

次設 ε_{ijk} 服從相關之多元常態分布，則

ε_{ijk} $(i=1,2\cdots p,\ j=1,2\cdots q,\ k=1,2\cdots r)$ 之聯合頻度函數爲

$$P=(2\pi)^{-\frac{pqr}{2}}\,|V|\,exp\left\{-\frac{1}{2}Q\right\}\qquad\cdots\cdots(26)$$

式中 $|V|=\begin{pmatrix}M & & 0\\ & \ddots & \\ 0 & & M\end{pmatrix}(r,r)$ 型　而 $M=\begin{pmatrix}\lambda^2+\sigma^2 & & -\frac{1}{pq-1}\lambda^2\\ & \ddots & \\ -\frac{1}{pq-1}\lambda^2 & & \lambda^2+\sigma^2\end{pmatrix}(pq,pq)$ 型

$$\lambda^2=\frac{1}{pqr}\sum_k\sum_l p^2{}_{l(k)},\quad \sigma^2=\frac{1}{pq}\sum_i\sum_j\sigma^2{}_{ij}\quad 而\ E(e^2{}_{ijkl})=\sigma^2{}_{ij}..$$

$Q=z'|V|^{-1}z$, 式中 $|V|^{-1}$ 爲 $|V|$ 之逆行列

$|V|^{-1}=\begin{pmatrix}M' & & 0\\ & \ddots & \\ 0 & & M'\end{pmatrix}(r,r)$ 型　$M'=\begin{pmatrix}\frac{1}{pq-1}\lambda^2+\sigma^2 & & \frac{1}{pq-1}\lambda^2\\ & \ddots & \\ \frac{1}{pq-1}\lambda^2 & & \frac{1}{pq-1}\lambda^2+\sigma^2\end{pmatrix}\Big/\left(\frac{pq}{pq-1}\lambda^2+\sigma^2\right)\sigma^2$ (pq,pq) 型

$$z=\begin{bmatrix}\varepsilon_{111}\\ \vdots\\ \varepsilon_{pq1}\\ \vdots\\ \varepsilon_{11r}\\ \vdots\\ \varepsilon_{pqr}\end{bmatrix}\begin{matrix}\left.\right\}pq\\ \vdots\\ \left.\right\}pq\end{matrix}\Bigg\}r$$

由 (26) 式爲基礎，利用最大可能比 (Likelihood ratio criteria) 方法可知

H_1, $(\alpha\beta)_{ij}=0$下, $F=\dfrac{SSA\times B}{(p-1)(q-1)}\Big/SSE/(pq-1)(r-1).$ $f_1=(p-1)(q-1)$, $f_2=(pq-1)(r-1)$

$$H_2,\ (\alpha)_i=0, \qquad F=\frac{SSA}{p-1}\Big/ SSE/(pq-1)(r-1) \qquad f_1=(p-1),\ f_2=(pq-1)(r-1)$$

$$H_3.\ (\beta)_j=0, \qquad F=SSB/q-1\Big/\frac{SSE}{(pq-1)(r-1)} \qquad f_1=q-1,\ f_2=(pq-1)(r-1),$$

其詳細之演算步驟可仿 Hatamura, Okuno & Sasaki, (1954)，茲不詳述。

八、ε_{jik} 不服從常態分布時之顯著性測定

在假設 (a) 之前提下，ε_{ijk} 不服從常態分布，則可利用 Pitman (1937) 之方法而求近似之 F 分布。爲簡便起見設 $\eta_{ijl(k)}=\eta_{ij(k)}=0,\ p_{l(k)}=0$ 且 $r_k=0$ 則機誤項之平方和成爲

$$\sum_i\sum_j\sum_k y^2{}_{ijk}-\frac{(y_{...})^2}{pqr}-\sum_i\sum_j\frac{y^2{}_{ij.}}{r}+\frac{y^2{}_{...}}{pqr}=SSE+SSR.$$

其自由度爲 $pq(r-1)=(pq-1)(r-1)+(r-1)$。此時本設計則成爲完全區集設計 (Completely Blocks)。

茲爲計算近似F分布起見，先求下列各期望值，並假定

$$E(e_{ijkl}{}^2)=\sigma^2{}_{ij}, \qquad \frac{1}{pq}\sum_i\sum_j\sigma^4{}_{ij}=\hat{\sigma}^4$$

$$\frac{1}{pq}\sum_i\sum_j\sigma^2{}_{ij}=\sigma^2 \quad 而 \quad \sigma^4=\hat{\sigma}^4.$$

$$E(\varepsilon^4{}_{ijk})=\tau^4{}_{ij}, \quad \tau^4{}_{ij}=E(e_{ijkl}{}^4) \cdots\cdots(27)$$

$$E(\varepsilon^3{}_{ijk}\ \varepsilon_{i'j'k'})=0 \cdots\cdots(28)$$

$$E(\varepsilon^2{}_{ijk}\ \varepsilon^2{}_{i'j'k'})=\sigma^2{}_{ij}\sigma^2{}_{i'j'}. \cdots\cdots(29)$$

故 $E[\sum\limits_{i\neq i'}\sum\limits_j\sum\limits_k \varepsilon^2{}_{ijk}\,\varepsilon^2{}_{i'jk}]=pqr(p-1)\hat{\sigma}^4$

$E[\sum\limits_i\sum\limits_{j\neq j'}\sum\limits_k \varepsilon^2{}_{ijk}\,\varepsilon^2{}_{ij'k}]=pqr(q-1)\hat{\sigma}^4$

$E[\sum\limits_{i\neq i'}\sum\limits_{j\neq j'}\sum\limits_k \varepsilon^2{}_{ijk}\,\varepsilon^2{}_{i'j'k}]=pqr(p-1)(q-1)\hat{\sigma}^4$

$E[\sum\limits_{i\neq i'}\sum\limits_j\sum\limits_{k\neq k'} \varepsilon^2{}_{ijk}\,\varepsilon^2{}_{i'jk'}]=pqr(p-1)(r-1)\hat{\sigma}^4.$

$E[\sum\limits_{i\neq i'}\sum\limits_{j\neq j'}\sum\limits_{k\neq k'} \varepsilon^2{}_{ijk}\,\varepsilon^2{}_{i'j'k'}]=pqr(p-1)(q-1)(r-1)\hat{\sigma}^4.$

餘類推

$$E(\sum_i\sum_{i'}\sum_{i''}\sum_j\sum_{j'}\sum_{j''}\sum_k\sum_{k'}\sum_{k''}\varepsilon^2{}_{ijk}\,\varepsilon_{i'j'k'}\,\varepsilon_{i''j''k''})=0. \cdots\cdots(30)$$

$$E(\sum_i\sum_{i'}\sum_{i''}\sum_{i'''}\sum_j\sum_{j'}\sum_{j''}\sum_{j'''}\sum_k\sum_{k'\neq k''}\sum_{k'''}\varepsilon_{ijk}\varepsilon_{i'j'k'}\varepsilon_{i''j''k''}\varepsilon_{i'''j'''k'''})=0 \cdots\cdots(31)$$

餘類推

但 $SSA+SSA\times B=\sum\limits_i\sum\limits_j\frac{y^2{}_{ij.}}{r}-\sum\limits_j\frac{y^2{}_{.j.}}{pr}$

故
$$E\Big[\sum_i\sum_j\frac{\varepsilon^2{}_{ij.}}{r}-\sum_j\frac{\varepsilon^2{}_{.j.}}{pr}\Big]^2=E\Big[\frac{1}{r}\Big(1-\frac{1}{p}\Big)\sum_i\sum_k\sum_j\varepsilon^2{}_{ijk}$$
$$+\frac{1}{r}\Big(1-\frac{1}{p}\Big)\sum_i\sum_j\sum_{k\neq k'}\varepsilon_{ijk}\,\varepsilon_{ijk'}-\frac{1}{pr}\sum_j\sum_{i\neq i'}\sum_k\varepsilon_{ijk}\,\varepsilon_{i'jk}$$
$$-\frac{1}{pr}\sum_j\sum_{i\neq i'}\sum_{k\neq k'}\varepsilon_{ijk}\,\varepsilon_{i'jk'}\Big]^2$$

展開並代入以上各關係式，則得

$$\varepsilon_{ijk}\{E(SSA+SSA\times B)^2\}=\frac{1}{pr}(p-1)^2q[\tau^4+(pqr-1)\hat{\sigma}^4]+\frac{1}{pr}2(p-1)(rp-r-p)\hat{\sigma}^4$$

故在 $(\alpha\beta)_{ij}=(\alpha)_i=0$ 之擬說下

$$V_{(SSA+SSA\times B)}=E\{(SSA+SSA\times B)^2-[E(SSA+SSA\times B)]^2\}$$

但 $E(SSA+SSA\times B)=q(p-1)\sigma^2.$

故如 $\sigma^4=\hat{\sigma}^4$，則

$$V_{(SSA+SSA\times B)}=\frac{1}{pr}(p-1)^2q(\tau^4-\hat{\sigma}^4)+\frac{1}{pr}2(p-1)q(rp-r-p)\hat{\sigma}^4. \qquad (32)$$

同理得；(因 $r=0$, 由假定)

$$V[_{(SSA+SSA\times B+(SSR+SSE)}]=\frac{1}{pr}(pr-1)^2q(\tau^4-\hat{\sigma}^4)+\frac{1}{pr}2q(2rp-r-p)\hat{\sigma}^4. \qquad (33)$$

$$\begin{aligned}&\mathrm{Cov}\,(SSA+SSA\times B,\ SSA+SSA\times B+(SSR+SSE))\\&=E[(SSA+SSA\times B)(SSA+SSA\times B+SSR+SSE)]-E(SSA+SSA\\&\quad\times B)\times E(SSA+SSA\times B+SSR+SSE)\\&=\frac{1}{pr}(pr-1)(p-1)q(\tau^4-\hat{\sigma}^4)+\frac{1}{pr}q(p-1)r\,\hat{\sigma}^4. \qquad (34)\end{aligned}$$

但由“比”之估算（見統計學辭典 PP355–357)

$$\begin{aligned}&E\left(\frac{SSA+SSA\times B}{SSA+SSA\times B+SSR+SSE}\right)\doteq\frac{E(SSA+SSA\times B)}{E(SSA+SSA\times B+SSR+SSE)}\\&\quad\left\{1+\frac{V(SSA+SSA\times B+SSR+SSE)}{E(SSA+SSA\times B+SSR+SSE)}\right.\\&\quad\left.-\frac{\mathrm{Cov}(SSA+SSA\times B,\ SSA+SSA\times B+SSR+SSE)}{E(SSA+SSA\times B)\times E(SSA+SSA\times B+SSR+SSE)}\right\}\\&\quad\doteq\frac{p-1}{pr-1}\left[1-\frac{2p(r-1)^2}{pqr(pr-1)^2}\right]=A \qquad (35)\end{aligned}$$

$$\begin{aligned}&V\left(\frac{SSA+SSA\times B}{SSA+SSA\times B+SSR+SSE}\right)\doteq\left(\frac{E(SSA+SSA\times B)}{E(SSA+SSA\times B+SSR+SSE)}\right)^2\\&\quad\times\left[\frac{V(SSA+SSA\times B)}{\{E(SSA+SSA\times B)\}^2}+\frac{V(SSA+SSA\times B+SSR+SSE)}{\{E(SSA+SSA\times B+SSR+SSE)\}^2}\right.\\&\quad\left.-2\frac{\mathrm{Cov}(SSA+SSA\times B,\ SSA+SSA\times B+SSR+SSE)}{E(SSA+SSA\times B)\times E(SSA+SSA\times B+SSR+SSE)}\right]\\&\quad\doteq\frac{2(p-1)}{pr-1}\left[\frac{(p-1)(r-1)(pr-3)}{pqr(pr-1)^2}+O\left(\frac{1}{p^2qr^2}\text{，或 }\frac{1}{pqr^3}\text{ 等}\right)\right]\\&\quad\doteq\frac{2(p-1)^2(r-1)(pr-3)}{pqr(pr-1)^3}=B. \qquad (36)\end{aligned}$$

其次設 $\dfrac{(SSA+SSA\times B)/q(p-1)}{[(SSR+SSE)]/r(pq-1)}$ 在 $r_k=0$, $p_{l(k)}=0$, $\eta_{ijl(k)}=0$ 之下，及擬說 $(\alpha)_i=(\alpha\beta)_{ij}=0$ 時成爲F分布。

則 $(SSA+SSA\times B)/(SSA+SSA\times B+SSR+SSE)$ 成爲 Beta 分布。

其自由度近似爲

$$f_1=\frac{2A[A(1-A)-B]}{B}\text{，}\qquad f_2=\frac{2(1-A)[A(1-A)-B]}{B} \qquad (37)$$

式中 $A=\dfrac{p-1}{pr-1}\left[1-\dfrac{2p(r-1)^2}{pqr(pr-1)^2}\right]$

$$B=\frac{2(p-1)^2(r-1)(pr-3)}{pqr(pr-1)^3}$$

同理對於 $(\beta)_j=(\alpha\beta)_{ij}=0$ 之擬說亦可如上進行；而得在 $\sigma^4=\hat{\sigma}^4$, $r_k=0$, $p_{l(k)}=0$. $\eta_{ijl(k)}=\eta_{ij(k)}=0$ 下，$\dfrac{SSB+SSA\times B}{SSB+SSA\times B+(SSR+SSE)}$ 近似成爲 Beta 分布。

$$f_1=\frac{2A'[A'(1-A')-B']}{B'},\quad f_2=\frac{2(1-A')[A'(1-A')-B']}{B'} \cdots\cdots\cdots\cdots\cdots\cdots(38)$$

而 $A'=\dfrac{q-1}{qr-1}\left(1-\dfrac{2q(r-1)^2}{pqr(qr-1)^2}\right).$

$$B'=\frac{2(q-1)^2(r-1)(qr-3)}{pqr(qr-1)^3}.$$

而 $E\left(\dfrac{SSB+SSA\times B}{SSB+SSA\times B+SSR+SSE}\right)\doteq\dfrac{q-1}{qr-1}\left(1-\dfrac{2q(r-1)^2}{pqr(qr-1)^2}\right)\cdots\cdots\cdots(39)$

$$V\left(\frac{SSB+SSA\times B}{SSB+SSA\times B+SSR+SSE}\right)\doteq\frac{2(q-1)^2(r-1)(qr-3)}{pqr(qr-1)^3}.$$

九、各效應內各水準間之比較與單一自由度劃分

在假設 (c) 之前提下，即 $z_{ijkl}\ \varepsilon\mathrm{NID}(0,\sigma^2)$，且 $r_l=0$, $\eta_{ijl(k)}=\eta_{ij(k)}=0$ 則 (3) 之數理模型成爲

$$\begin{aligned}y_{ijk}=&\mu+0\cdot\alpha_1+0\cdot\alpha_2+\cdots1\cdot a_i+\cdots0\cdot a_p+0\cdot b_1+0\cdot b_2+\cdots+1\cdot b_j\\&+\cdots0\cdot b_q+0\cdot(ab)_{11}+0\cdot(ab)_{12}+0\cdot(ab)_{13}+\cdots1\cdot(ab)_{ij}+\cdots0\cdot(ab)_{pq}+0\cdot r_1\\&+0\cdot r_2+\cdots1\cdot r_k+\cdots0\cdot r_r+0\cdot e_{111}+0\cdot e_{112}+\cdots1\cdot e_{ijk}+\cdots+0\cdot e_{pqr}\ \cdots\cdots\cdots(10)\end{aligned}$$

玆以行列代數表之則爲

$$y=x_1a+x_2\beta+x_3R+e=x\beta+e.\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(11)$$

式中 $x\beta=x_1a+x_2\beta+x_3R=(x_1,x_2,x_3)\begin{bmatrix}a\\b\\R\end{bmatrix}$

$$x=(x_1,x_2,x_3),\quad \beta=\begin{bmatrix}a\\b\\R\end{bmatrix}$$

玆設 $\hat{\beta}$ 爲 β 之估值，$xx'=S$，則得

$S\hat{\beta}=x'y$ 而 $E(x'e)=0$, $E(\hat{\beta})=B$. (見 Kempthorne 1952 pp. 74–78)

由此代入以上關係得到下列 5 組聯立方程式。

$$\left.\begin{aligned}
&pqr\hat{\mu}+qr\hat{a}_1+qr\hat{a}_2+\cdots qr\hat{a}_p+pr\hat{b}_1+pr\hat{b}_2+\cdots pr\hat{b}_q+r(ab)_{11}\\
&\quad+r(ab)_{12}+\cdots+r(ab)_{ij}+\cdots r(ab)_{pq}+pq\hat{r}_1+pq\hat{r}_2+\cdots pqr_r=Y_{\cdots}\ \mathrm{O}\\
&\left.\begin{aligned}
&qr\hat{\mu}+qr\hat{a}_1+r\hat{b}_1+r\hat{b}_2+\cdots r\hat{b}_q+r(ab)_{11}+r(ab)_{12}+\cdots r(ab)_{1q}+q\hat{r}_1\\
&\quad+q\hat{r}_2+\cdots q\hat{r}_r=Y_{1\cdot\cdot}\\
&qr\hat{\mu}+qr\hat{a}_2+r\hat{b}_1+r\hat{b}_2+\cdots r\hat{b}_q+r(ab)_{21}+r(ab)_{22}+\cdots r(ab)_{2q}+q\hat{r}_1\\
&\quad+q\hat{r}_2+\cdots q\hat{r}_r=Y_{2\cdot\cdot}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&qr\hat{\mu}+qr\hat{a}_p+r\hat{b}_1+r\hat{b}_2+\cdots r\hat{b}_q+r(ab)_{p1}+r(ab)_{p2}+\cdots r(ab)_{pq}+q\hat{r}_1\\
&\quad+q\hat{r}_2+\cdots q\hat{r}_r=Y_{p\cdot\cdot}
\end{aligned}\right\}\cdots\mathrm{I}\\
&\left.\begin{aligned}
&pr\hat{\mu}+r\hat{a}_1+r\hat{a}_2+\cdots r\hat{a}_p+pr\hat{b}_1+r(ab)_{11}+r(ab)_{21}+\cdots r(ab)_{p1}+p\hat{r}_1\\
&\quad+p\hat{r}_2+\cdots p\hat{r}_r=Y_{\cdot1\cdot}\\
&pr\hat{\mu}+r\hat{a}_1+r\hat{a}_2+\cdots r\hat{a}_p+pr\hat{b}_2+r(ab)_{12}+r(ab)_{22}+\cdots r(ab)_{p2}+p\hat{r}_1\\
&\quad+p\hat{r}_2+\cdots p\hat{r}_r=Y_{\cdot2\cdot}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&pr\hat{\mu}+r\hat{a}_1+r\hat{a}_2+\cdots r\hat{a}_p+pr\hat{b}_q+r(ab)_{1q}+r(ab)_{2q}+\cdots r(ab)_{pq}+p\hat{r}_1\\
&\quad+p\hat{r}_2+\cdots p\hat{r}_r=Y_{\cdot q\cdot}
\end{aligned}\right\}\cdots\mathrm{II}
\end{aligned}\right|$$

$$\left.\begin{array}{l}
\left.\begin{array}{l}
r\hat{\mu}+r\hat{a}_1+r\hat{b}_1+r(ab)_{11}+\hat{r}_1+\hat{r}_2+\cdots\hat{r}_r=\mathrm{Y}_{11\cdot}\\
r\hat{\mu}+r\hat{a}_1+r\hat{b}_2+r(ab)_{12}+\hat{r}_1+\hat{r}_2+\cdots\hat{r}_r=\mathrm{Y}_{12\cdot}\\
\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
r\hat{\mu}+r\hat{a}_1+r\hat{b}_q+r(ab)_{1q}+\hat{r}_1+\hat{r}_2+\cdots\hat{r}_r=\mathrm{Y}_{1q\cdot}\\
r\hat{\mu}+r\hat{a}_2+r\hat{b}_1+r(ab)_{21}+\hat{r}_1+\hat{r}_2+\cdots\hat{r}_r=\mathrm{Y}_{21\cdot}\\
r\hat{\mu}+r\hat{a}_2+r\hat{b}_2+r(ab)_{22}+\hat{r}_1+\hat{r}_2+\cdots\hat{r}_r=\mathrm{Y}_{22\cdot}\\
\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
r\hat{\mu}+r\hat{a}_2+r\hat{b}_q+r(ab)_{2q}+\hat{r}_1+\hat{r}_2+\cdots\hat{r}_r=\mathrm{Y}_{2q\cdot}\\
\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
r\hat{\mu}+r\hat{a}_p+r\hat{b}_q+r(ab)_{pq}+\hat{r}_1+\hat{r}_2+\cdots\hat{r}_r=\mathrm{Y}_{pq\cdot}
\end{array}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\mathrm{III}\\
\left.\begin{array}{l}
pq\hat{\mu}+q\hat{a}_1+q\hat{a}_2+\cdots q\hat{a}_p+p\hat{b}_1+p\hat{b}_2+\cdots p\hat{b}_q+(ab)_{11}+\cdot+(ab)_{pq}\\
\quad+pq\hat{r}_1=\mathrm{Y}_{\cdot\cdot1}\\
pq\hat{\mu}+q\hat{a}_1+q\hat{a}_2+\cdots q\hat{a}_p+p\hat{b}_1+p\hat{b}_2+\cdots p\hat{b}_q+(ab)_{11}+\cdots+(ab)_{pq}\\
\quad+pq\hat{r}_2=\mathrm{Y}_{\cdot\cdot2}\\
\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
pq\hat{\mu}+q\hat{a}_1+q\hat{a}_2+\cdots q\hat{a}_p+p\hat{b}_1+p\hat{b}_2+\cdots p\hat{b}_q+(ab)_{11}+\cdot+(ab)_{pq}\\
\quad+pq\hat{r}_r=\mathrm{Y}_{\cdot\cdot r}
\end{array}\right\}\cdots\mathrm{IV}
\end{array}\right\}\cdots(12)$$

設 $s\hat{\beta}$ 之一直線組合爲 $p's\hat{\beta}=\lambda'\hat{\beta}$，則

$$\lambda'\hat{\beta}=p'x'y=p'x'(x\hat{\beta}+e)=p's\hat{\beta}+p'x'e$$

故 $\lambda'\hat{\beta}$ 之變方爲

$$\begin{aligned}
\mathrm{E}(p'x'ee'xp)&=\sigma^2(p'x'xp)\\
&=\sigma^2(p'sp)\\
&=\sigma^2p'\lambda\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(13)
\end{aligned}$$

由此關係式，吾人玆求各效應間之比較及其變方如下所示；在此必須

$$\sum_i(ab)_{ij}=\sum_j(ab)_{ij}=0.$$

(1) $(ab)_{ij}-(ab)_{i'j'}\qquad ij\neq i'j'$

$$=\frac{1}{r}\left[\left(\mathrm{Y}_{ij\cdot}-\frac{\mathrm{Y}_{i\cdot\cdot}}{q}-\frac{\mathrm{Y}_{\cdot j\cdot}}{p}+\frac{\mathrm{Y}_{\cdots}}{pqr}\right)-\left(\mathrm{Y}_{i'j'}-\frac{\mathrm{Y}_{i'\cdot\cdot}}{q}-\frac{\mathrm{Y}_{\cdot j'\cdot}}{p}+\frac{\mathrm{Y}_{\cdots}}{pqr}\right)\right]$$

此處 $p'_{ab}=(0,\ \underbrace{0\cdots0\overset{i}{-\frac{1}{qr}},\ 0\cdots0\overset{i'}{\frac{1}{qr}},\ 0\cdots0}_{p}\ \underbrace{00\cdots\overset{j}{-\frac{1}{pr}}00\cdots\overset{j'}{\frac{1}{pr}}0\cdots0}_{q}\ \underbrace{00\cdots\overset{ij}{\frac{1}{r}}0\cdots0\overset{i'j'}{-\frac{1}{r}}0\cdots0}_{pq}\ \underbrace{00\cdots00}_{r})$

$$\lambda_{ab}=\begin{array}{r}\\ \\ \\ \\ \\ i\\ \\ \\ \\ i'\\ \\ \\ \\ \end{array}\left[\left.\begin{array}{c}0,\\0\\0\\\vdots\\0\\-1\\0\\\vdots\\0\\1\\0\\\vdots\\0\end{array}\right\}p\right]$$

故得

$$\begin{aligned}
&\mathrm{Var}[(ab)_{ij}-(ab)_{i'j'}]\\
&=p'_{ab}\lambda_{ab}\sigma^2\\
&=\left(\frac{1}{r}+\frac{1}{pr}+\frac{1}{qr}+\frac{1}{r}+\frac{1}{pr}+\frac{1}{qr}\right)\sigma^2\\
&=\frac{2}{r}\left(1+\frac{1}{p}+\frac{1}{q}\right)\sigma^2
\end{aligned}$$

$$\begin{array}{c} \\ \\ \\ j \\ \\ \\ \\ j' \\ \\ \\ \\ \\ \\ \\ ij \\ \\ \\ \\ i'j' \\ \\ \\ \\ \\ \\ \\ \end{array} \left. \begin{array}{c} \left.\begin{array}{c} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right\} q \\ \left.\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{array}\right\} pq \\ \left.\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} r \end{array} \right)$$

(2) $\hat{a}_i-\hat{a}_{i'}$

$$=\frac{1}{qr}(\mathrm{Y}_{i}..-\mathrm{Y}_{i'}..)$$

故

$$\mathrm{Var}(\hat{a}_i-\hat{a}_{i'})=p'_\alpha\lambda_\alpha\sigma^2=\frac{2}{qr}\sigma^2.$$

式中 $p'_\alpha=(0,\underbrace{00\cdots0\overset{i}{\frac{1}{qr}},0\cdots0-\overset{i'}{\frac{1}{qr}}00\cdots0}_{p}\ \underbrace{00\cdots00}_{q}\ \underbrace{00\cdots00}_{pq}\ \underbrace{00\cdots00}_{r})$

$$\lambda_\alpha=\begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ i' \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \left(\begin{array}{c} 0, \\ \left.\begin{array}{c} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ \vdots \\ 0 \end{array}\right\} p \\ \left.\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} q \\ \left.\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} pq \\ \left.\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} r \end{array} \right)$$

同理求得

(3) $\hat{b}_j-\hat{b}_{j'}$

$$=\frac{1}{pr}(\mathrm{Y}._{j}.-\mathrm{Y}._{j'}.)$$

而 $\mathrm{Var}(\hat{b}_j-\hat{b}_{j'})=\dfrac{2}{pr}\sigma^2$

此種方法可以推至單一自由度之劃分。此處A試因有 $(p-1)$ 個自由度，可區分爲一個直線效應自由度，一個二次效應自由度，……及 $(p-1)$ 次效應之一個自由度。同理B試因 $(q-1)$ 個自由度可區分爲一個直線效應自由度，一個二次效應自由度，……及 $(q-1)$ 次效應之一個自由度。互交之 $(p-1)(q-1)$ 個自由度亦可同樣區分爲A試因直線效應與B試因直線效應互作一個自由度，……A試因直線效應與B試因二次效應互作一個自由度，……以此類推共計 $(p-1)(q-1)$ 個單一自由度。

此種單一自由度之區分，可利用直交多項式之均衡尺度 (Scale of orthogonal polynomial) 進行之，此等直交係數則可利用迴歸方法求得。故每一單一自由度對應於其中一個尺度。在上之 $p\times q$ 複因子試驗中，A試因 $p-1$ 個自由度之區分可利用式中之 (I) 進行。B試因 $q-1$ 個自由度之區分可利用式中之 (II) 進行。$\mathrm{A}\times\mathrm{B}$ 之 $(p-1)(q-1)$ 個自由度則

可利用式中之 (III) 而進行之。茲考慮A試因單一自由度之區分，令其第 i 級效應一個自由度之均衡尺度為 $S_{i1}, S_{i2}, \cdots S_{ip}$。第 i' 級效應一個自由度之均衡尺度為 $S_{i'1}, S_{i'2}, \cdots S_{i'p}$。

則顯然可知 $\sum_{j=1}^{p} S_{ij} = \sum_{j=1}^{p} S_{i'j} = 0$，且 $\sum_{j} S_{ij} S_{i'j} = 0$ $(i \neq i')$。第 i 級效應之效果則為

$$\sum_j s_{ij}\alpha_j = \frac{1}{qr}\left(\sum_j s_{ij}\, Y_{j\cdot\cdot}\right) = \left(\frac{s_{ij}}{qr}\right)' \cdot X'y.$$

式中 $$\left(\frac{S_{ij}}{qr}\right)' = (0, \underbrace{\frac{S_{i1}}{qr}, \frac{S_{i2}}{qr}, \cdots \frac{S_{ip}}{qr}}_{p}, \underbrace{0\cdots0}_{q}, \underbrace{0\cdots0}_{pq}, \underbrace{0\cdots0}_{r}) = A'$$

而 $$X'y = \left[\begin{array}{c} y_{\cdots} \\ y_{1\cdot\cdot} \\ y_{2\cdot\cdot} \\ \vdots \\ y_{p\cdot\cdot} \\ y_{\cdot 1\cdot} \\ y_{\cdot 2\cdot} \\ \vdots \\ y_{\cdot q\cdot} \\ y_{11\cdot} \\ y_{12\cdot} \\ \vdots \\ y_{pq\cdot} \\ y_{\cdot\cdot 1} \\ y_{\cdot\cdot 2} \\ \vdots \\ y_{\cdot\cdot r} \end{array}\right]$$

因 $y = X\beta + e$.

故其變方為

$$E(A'X'ee'XA) = \sigma^2 (A'X'XA) = \sigma^2 A'SA$$
$$= \frac{1}{qr}\left(\sum_j S^2_{ij}\right)\sigma^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (14)$$

但 $$SA = \left[\begin{array}{c} 0 \\ \left.\begin{array}{c} S_{i1} \\ S_{i2} \\ \vdots \\ S_{ip} \end{array}\right\} p \\ \left.\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} q \\ \left.\begin{array}{c} \left.\begin{array}{c} \frac{1}{q}S_{i1} \\ \vdots \\ \frac{1}{q}S_{i1} \end{array}\right\} \\ \left.\begin{array}{c} \frac{1}{q}S_{i2} \\ \vdots \\ \frac{1}{q}S_{i2} \end{array}\right\} \\ \vdots \\ \left.\begin{array}{c} \frac{1}{q}S_{ip} \\ \vdots \\ \frac{1}{q}S_{ip} \end{array}\right\} \end{array}\right\} pq \\ \left.\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right\} r \end{array}\right]$$

故得A試因第 i 效應之變方為

$$\left(\sum_j S_{ij}\frac{Y_{j\cdot\cdot}}{qr}\right)^2 \Bigg/ \left(\sum_j \frac{S^2_{ij}}{qr}\right)$$

對於其他各效應亦可同樣推得。

欲進行顯著性測驗，可利用F或 t。如欲測驗第 i 級效應與 0 是否有顯著差異，則利用下列 t 值

$$t = \left(\sum_j S_{ij}\frac{Y_{j\cdot\cdot}}{qr}\right) \Bigg/ \sqrt{\sum_j S^2_{ij}}\ \sqrt{\sigma/qr}$$

自由度為 $(pq-1)(r-1)$

σ^2 由 SSE/$(pq-1)(r-1)$ 估算。其危險域爲 $t_{0.05} \leqq |t|$。其在 0.95 下之信賴限界 (Confidence limit) 爲

$$\sum S_{ij}\frac{Y_{j\cdot\cdot}}{qr} \pm t_{0.05,\ (pq-1)(r-1)}\sqrt{\sum_j S^2_{ij}}\ \sqrt{\frac{SSE}{qr(qr-1)(r-1)}}$$

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ON THE ANALYSIS OF P×Q FACTORIAL EXPERIMENT

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1. In a paper "On the design and analysis of field experiments" Hatamura, Okuno and Sasaki (1954) discussed the randomization models and the concerning likelihood ratio tests of the completely randomized design and the randomized and the randomized block design. A similar treatise on these subjects was also presented previously in a famous book of Prof. Kempthorne, "The design and analysis of experiments", which was published in 1952. It seems to the author that this procedures can similarly be applied to the analysis of P×Q factorical experiment. With this in mind the present paper is directed to provide a systematic presentation of this generalized factorical experiment. Among the topics discussed emphases are placed on the construction of mathematical model and the randomization test, the sums of squares for each effective factor and their expectations when randomization is under examination, and the assumptions and the related tests of hypotheses. Besides, the writer considers the single degrees of freedom in detail, borrowing the method of matrix notation from Kemthorne's book, pp 74–79.

2. The mathematical model of P×Q factorical experiment, composed of r replication each with pq plots, can be formulated as follows:

$$y_{ijkl}=u+a_i+b_j+(ab)_{ij}+r_k+(p_{l(k)}+n_{ijl(k)}+e_{ijkl})$$
$$=u+a_j+b_j+(ab)_{ij}+r_k+z_{ijkl}$$
$$\sum_i a_i=0,\ \sum_j b_j=0,\ \sum_i(ab)_{ij}=\sum_j(ab)_{ij}=0,\ \sum_k r_k=0,\ \sum_l p_{l(k)}=0$$
$$\sum_i\sum_j n_{ijl(k)}=\sum_l n_{ijl(k)}=0$$

where y_{ijkl}=the yield of (ij) combination in lth plot of kth block.

u=over all mean effect. a_i=the effect of the ith level of factor A. b_j=the effect of the jth level of factor B. $(ab)_{ij}$=the interaction effect of the ith level of factor A with the jth level of factor B. r_k=the effect of kth replicate. $p_{l(k)}$=the effect of lth plot in kth block. $n_{ijl(k)}$=the interaction effect of the (ij)th combination with the lth plot in kth block, and e_{ijkl}=a random

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component of error associated with the (ij)th combination in lth plot of kth block. Here in this paper the writer associates the interaction effect between (ij)th combination and the kth block with the random variate e_{ijkl}.

$l$et ε_{ijkl} denote the sum of $p_{l(k)}$, $n_{ijl(k)}$ and e_{ijkl} and $l$et it be considered as the error term attached to y_{ijkl}, then we have two kinds of errors. One is systematic, composed of $p_{l(k)}$ and $n_{ijl(k)}$, and the other is random, caused by e_{ijkl}. From here, we see further that, associated with the quantity ε_{ijkl} the randomization comes into action. For the investigation of this effect, the writer introduced a new variate δ^l_{ijk} whose distribution is specified by randomization, as given in Kempthorrne' book, where

$\delta^l_{ijk}=1$ when the (ij)th combination is allocated to the lth plot in kth block.

$=0$, otherwise.

The probabilities and expectations associated with δ^l_{ijk} are then specified as in (5) & (6). With this we further have (Randomizotion Model)

$$y_{ijk}=u+a_i+b_j+(ab)_{ij}+r_k+\sum_l \delta^l_{ijk}\, p_{l(k)}+\sum_l \delta^l_{ijk}\, n_{ijl(k)}+\sum_l \delta^l_{ijk},\, e_{ijkl}$$

where y_{ijk}=the yield of (ij)th combination in the kth block.

3. The sums of squares attached to each effect are obtained by method of least square and then the expectations of mean squares are derived for each terms respectively. Thus we have:

For block:

$$E\left(\frac{SSR}{(r-1)}\right)=pq\left(\frac{1}{r-1}\sum_k r^2{}_k\right)+\sigma^2+\frac{1}{pqr(pq-1)}\sum_{i,}\sum_{j,}\sum_{k,}\sum_l (n_{ijl(k)})^2$$

For A factor:

$$E\left(\frac{SSA}{(p-1)}\right)=pq\left(\frac{1}{p-1}\sum_i a^2{}_i\right)+\sigma^2+\frac{1}{pq-1}\sum_l p^2{}_{l(k)}$$
$$+\frac{1}{pqr(pq-1)}\sum_{i,}\sum_{j,}\sum_{k,}\sum_l (n_{iijl(k)})^2-\frac{q}{(p-1)pqr(pq-1)}\sum_i\sum_k\sum_l \eta^2{}_{i\cdot l(k)}$$

For B factor:

$$E\left(\frac{SSB}{(q-1)}\right)=pr\left(\frac{1}{q-1}\sum_j b^2{}_j\right)+\sigma^2+\left(\frac{1}{pq-1}\right)\sum_l p^2{}_{l(k)}$$
$$+\frac{1}{pqr(pq-1)}\sum_{i,}\sum_{j,}\sum_{k,}\sum_l (n_{ijl(k)})^2-\frac{p}{(q-1)pqr(pq-1)}\sum_j\sum_k\sum_l \eta^2{}_{\cdot j^l(k)}$$

For the interaction A×B:

$$E\left(\frac{SSA\times B}{(p-1)(q-1)}\right)=r\left(\frac{1}{(p-1)(q-1)}\sum_{i,}\sum_j (ab)^2{}_{ij}\right)+\sigma^2$$
$$+\frac{1}{pq-1}\sum_l p^2{}_{l(k)}+\frac{1}{pqr(pq-1)}\sum_{i,}\sum_{j,}\sum_{k,}\sum_l (n_{ijl(k)})^2$$
$$-\frac{1}{(p-1)(q-1)pqr(pq-1)}\sum_i\sum_j\sum_k\sum_l (\eta_{ijl(k)}-\eta_{i\cdot l(k)}-\eta_{\cdot jl(k)})^2$$

For the error term:

$$\mathrm{E}\left(\frac{\mathrm{SSE}}{(pq-1)(r-1)}\right)=\sigma^2+\frac{1}{pq-1}\sum_l p^2{}_{l(k)}+\frac{pq-2}{(pq-1)pqr(pq-1)}\sum_{i}\sum_{j}\sum_{k}\sum_{l}(n_{ijl(k)})^2$$

where $\sigma^2=\frac{1}{(pq)^2r}\sum_{i}\sum_{j}\sum_{k}\sum_{l}\sigma^2{}_{ijkl}$

Here we have additional terms concerning the effect of plot within block and interaction between combination and plot. The tests of hypothesis for each factor and the interaction between them are therefore specified by assumptions made for ε_{ijk} in connection with these additional components. The assumptions concerned are as follows, (produced by Hatamura *et al* in randomized block design).

Assumption 1: The variate ε_{ijk} has the 1st, 2nd, 3rd and 4th moments for every $e\cdot\varepsilon_{ijk}$ does not necessarily follow the normal distribution.

Assumption 2: $\{\varepsilon_{ijkl};\ i=1,2..p.\ j=1,2..q.\ k=1,2..r.\}$ is a set of multivariate normally distributed random variables with $n_{ijl(k)}=0$ and $\sigma_{ijl}=\sigma$.

Assumption 3: $\varepsilon_{ijk}\ \varepsilon\mathrm{NID}(0,\sigma^2)$, $\sigma_{ijl}=\sigma$ and $n_{ijl(k)}=0$.

4. Now we are setting up the following three hypotheses in connection with factor A and B and the interaction between them: (Under randomization Model)

H_0: $(ab)_{ij}=0$, for $i=1,2..p.\ j=1,2..q.$

H_1: $a_i=0$, for $i=1,2..p.$

H_2: $b_j=0$, for $j=1,2..q.$

Under the assumption III it can be shown by the method of likelihood ratio test that it is sufficient to follow the following three F-tests for the corresponding test of hypotheses:

$\mathrm{F}_1=\frac{\mathrm{SSA\times B}/(p-1)(q-1)}{\mathrm{SSE}/(pq-1)(r-1)}$ to test H_0 with $df_1=(p-1)(q-1)$, $df_2=(pq-1)(r-1)$.

$\mathrm{F}_2=\frac{\mathrm{SSA}/(p-1)}{\mathrm{SSE}/(pq-1)(r-1)}$ to test H_1 with $df_1=(p\text{-}1)$, $df_2=(pq\text{-}1)(r\text{-}1)$.

$\mathrm{F}_3=\frac{\mathrm{SSB}/(q-1)}{\mathrm{SSE}/(pq-1)(r-1)}$ to test H_2 with $df_1=(q\text{-}1)$, $df_2=(pq\text{-}1)(r\text{-}1)$.

Under the assumption II, the F-tests are the same as those performed under assumption III. This can be shown to be the case by method of likelihood ratio test as is for assumption III.

Under the assumption 1, the method of fitting moment by Pitman (1937) is exercised, extending the derivation of Hatamura *et al* in randomized block design. For simplicity we assume $r_k=0$, and consider the hypothesis $(\alpha)_i=(\alpha\beta)_{ij}=0$ or $(\beta)_j=(\alpha\beta)_{ij}=0$. Then it can be expected that the necessary condition for the following ratio estimates r_1 & r_2 to approximate the corresponding Beta-distributions is

$\sigma^2{}_{ij}=\sigma^2=$constant$\rightarrow\hat{\sigma}^4=\sigma^4$, provided that $n_{ijl(k)}=n_{ij(k)}=0$, $p_{l(k)}=0$ and $\varepsilon_{ijl(k)}$ differs very little from the normal distribution, where

$$\sigma^2=\frac{1}{pq}\sum_i^p\sum_j^q e^2{}_{ij},\quad \mathrm{E}[(e_{ijkl})^2]=\sigma^2{}_{ij},\quad \hat{\sigma}^4=\frac{1}{pq}\sum_i^p\sum_j^q\sigma^4{}_{ij}.$$

$$r_1=\frac{\mathrm{SSA+SSA\times B}}{\mathrm{SSA+SSA\times B+(SSR+SSE)}}$$

$$r_2=\frac{\text{SSB}+\text{SSA}\times\text{B}}{\text{SSB}+\text{SSA}\times\text{B}+(\text{SSR}+\text{SSE})}$$

The f_1 & f_2's are specified as in (37) and (38).

5. In order to lead the comparison of each factor and the single degrees of freedom here the author borrowed the method of matrix notation from Kempthorne. Thus we have the set of equations in (12) by expanding the expression $SB=Xy$. The variances for the comparisons were then obtained as $2/r\ (1/r+1/p+1/q)\sigma^2$, $2/qr\sigma^2$ and $2/pr\sigma^2$ for factors A, B and the interaction between them respectively under the premise of $\sum_i(ab)_{ij}=\sum_j(ab)_{ij}=0$. The partition of degrees of freedom of each factor is obtained with the scales of orthogonal polynomial which was given in Snedecor (1946). Let s_{ij}, $j=1,2..p$ be the scale associated with the ith effect of factor A. Then we have, at the estimate of this effect, the following expression:

$$\sum_j s_{ij}a_j=\frac{1}{qr}(\sum_j s_{ij}\ Y_{i..})=\left(\frac{(s_{ij})}{qr}\right)'.X'y=A'X'y \text{ in matrix notation}$$

$$\text{where } A'=\left(\frac{(s_{ij})}{qr}\right)'=(0,\ s_{i1}/qr,\ s_{i2}/qr,\cdots s_{ip}/qr,\ \underbrace{0\ldots.0}_{q},\ \underbrace{0\ldots.0}_{pq},\ \underbrace{0\ldots.0}_{r})$$

The variance of this comparison is obtained as

$$E(A'X'ee'XA)=\sigma^2A'SA=\frac{1}{qr}(\sum_j s^2_{ij})\sigma^2$$

where $S=XX'$ and therefore for the ith effect of Factor A we have as an estimate of σ^2; $\left(\sum_j s_{ij}\frac{Y_{j..}}{qr}\right)^2\Big/\left(\sum_j\frac{s^2_{ij}}{qr}\right)$

This can similarly be practiced for factor B and the interaction between A and B. (Summary).

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