THE ANALYSIS OF ABIB DESIGN IN THE SAMPLING THEORY FRAMEWORK[1]

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I. Introduction

Suppose that we have a BIB design with parameters B, v, r, k and λ. Suppose further that we have m additional control treatments and that we want to compare these control treatments with those given in the BIB design. If we add these m control treatments to each block of the BIB, then we obtain a new design which also has B blocks but each block has $k+m$ plots. In essence this design is a BIB combined with an randomized complete block design. We shall call this new design ABIB.

In this paper we shall assume a mixed model (Model III) and try to obtain the linear, unbiased and minimum varianced estimators (call it LUMV for short) of the treatment effects in the sampling theory framework. The corresponding results in the fixed model case (Model I) will be obtained from those of Model III by setting $w'=0$, the meaning of which will be given in section III.

II. Mathematical Model

The mathematical model may be written as

$$Y_{ij}=\mu+a_i+b_j+e_{ij} \tag{2.1}$$
$$i=1,2,\ \ldots\ldots v,\ v+1,\ \ldots\ldots v+m.$$
$$j=1,2,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots B.$$

where μ is the unknown population mean; Y_{ij} is the observed value of the i th treatment in the j th block; a_i, $i=1,2,\ldots.v$, $v+1,\ldots.v+m$, is the i th treatment effect ($i=1,2,\ldots.v$. for the BIB portion and $i=v+1\ldots.v+m$. for the other portion which corresponds to a randomized complete block design; the treatments labeled by $i=v+1,\ldots.v+m$ are called augmented treatments); b_j, $j=1,2,\ldots.$B, is the effect of the j th block; e_{ij} is the error attached to Y_{ij}.

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As usual, it is assumed that e_{ij} is identically and independently distributed with mean 0 and variance σ^2. In matrix notation we may write (2.1) as:

$$\underset{\sim}{Y}=l_N\mu+A\underset{\sim}{a}+B\underset{\sim}{b}+\underset{\sim}{e}, \tag{2.2}$$

where $\underset{\sim}{Y}$ is the N×1 (N=(k+m)B) column vector of observations; $\underset{\sim}{e}$ is the N×1 colum vector of errors; l_N is the N×1 column vector of which each component is unity; $\underset{\sim}{a}_{(v+m)\times 1}$ and $\underset{\sim}{b}_{B\times 1}$ are the column vector of the a_i 's and the b_j 's, respectively. Let the components of $\underset{\sim}{Y}$ be arranged first by order of the block subscripts and then by that of the treatment subscripts, with the m augmented treatments in each block being placed at the tail part of each sub-vector of $\underset{\sim}{Y}$ which corresponds to the subscript of that block. Then

$$B=\begin{pmatrix} l_{k+m} & & \underset{\sim}{0} \\ & \ddots & \\ \underset{\sim}{0} & & l_{k+m} \end{pmatrix}=l_{k+m}*I_B$$

$$A=\begin{pmatrix} A_1 \\ \vdots \\ A_B \end{pmatrix}$$

with

$$A_j=\begin{pmatrix} \delta_{[(k+m)(j-1)+1],1} & \cdots & \delta_{[(k+m)(j-1)+1],v}, & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \delta_{[(k+m)(j-1)+k],1} & \cdots & \delta_{[(k+m)(j-1)+k],v}, & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where

$$\delta_{st}=\begin{cases} 1, & \text{if the } s \text{ th observation associates with the } t \text{ th treatment.} \\ 0, & \text{otherwise} \end{cases}$$

$$s=1,2, \ldots N, \qquad t=1,2, \ldots (v+m)$$

In this paper we shall assume that the a_i 's are constants and $\underset{\sim}{b}\sim N(\underset{\sim}{0}, I_B\sigma_b^2)$, and further that the b_j 's are distributed independently of the e_{ij} 's. With this convention (2.1) then becomes the usual Model III. Before we proceed with our analysis, it should be remarked that Y_{ij}, $i=1,..v+m$, $j=1,2, \ldots$ B may or may not exist, depending on whether or not the i th treatment appears in the j th block. To allow for the effect of incomplete block we introduce

$$n_{ij}=\begin{cases}1, & \text{if the } i \text{ th treatment has appeared in the } j \text{ th block,}\\ 0, & \text{otherwise}\end{cases}$$

$$i=1,2,\ \ldots\ldots,\ v+m, \qquad j=1,2,\ldots\ldots,\ B$$

The matrix $N=(n_{ij})_{(v+m)\times B}$ is usually called the incidence matrix. The following relationships can be easily shown to hold:

$$Al_{(v+m)}=l_N,$$

$$l'_N A=(rl'_v,\ Bl'_m)$$

$$A'A=\begin{pmatrix} rI_v & \underline{0} \\ \underline{0} & BI_m \end{pmatrix}$$

$$A'B=N$$

$$Nl_B=\begin{pmatrix} rl_v \\ Bl_m \end{pmatrix}$$

$$l'_{v+m}N=(k+m)l_B$$

and

$$NN'=\begin{pmatrix} (r-\lambda)I_v+\lambda l_v l'_v & rl_v l'_m \\ rl_m l'_v & Bl_m l'_m \end{pmatrix}$$

III. Analysis under Model III

Under Model III we now try to obtain the LUMV estimators of the a_i's, $i=1,2,\ \ldots\ldots\ v+m$, in the sampling theory framework.

Under Model III, we have

$$E(\underline{Y})=\mu l_N+A\underline{a}, \qquad N=B(k+m)$$

$$V(\underline{Y})=\sigma_a^2(l_{k+m}\ l'_{k+m})*I_B+\sigma^2 I_{B(k+m)}$$

The LUMV estimator of $\underline{a}$ can then be obtained by using the Aitkin's singular generalized least square method.

Define now:

$$\left.\begin{aligned} & T_i=\sum_{j=1}^{B} n_{ij}Y_{ij}, && B_j=\sum_{i=1}^{v+m} n_{ij}Y_{ij} \\ & Q_i=T_i-\frac{1}{k+m}\sum_{j=1}^{B} n_{ij}B_j, \text{ and} && G=\sum_{j=1}^{B} B_j \\ & Q'_i=T_i-Q_i-\frac{r_i}{N}G && \\ & i=1,2,\cdots\cdots v+m, && j=1,2,\ \ldots.\ B \end{aligned}\right\} \qquad (3.1)$$

Then it can be easily shown that the LUMV estimator $\hat{\underline{a}}$ of $\underline{a}$ satisfies,

$$F\hat{\underline{a}}=\underline{P} \qquad (3.2)$$

where P is a $(v+m)\times 1$ column vector whose i th component P_i, $i=1,2,\ldots.v+m$, is:

$$P_i=\frac{1}{\sigma^2}Q_i+\frac{1}{\sigma^2+(k+m)\sigma_a^2}Q'_i=wQ_i+w'Q'_i$$

with $\quad w=\frac{1}{\sigma^2} \quad$ and $w'=\frac{1}{\sigma^2+(k+m)\sigma_a^2}$

The matrix $F_{(v+m)\times(v+m)}$ is given by

$$F=\begin{pmatrix}\frac{w(rm+\lambda_v)+w'(r-\lambda)}{k+m}I_v+\frac{1}{B(k+m)}[w'(\lambda B-r^2)-\lambda Bw]l_vl'_v, & \left(-\frac{r}{k+m}\right)wl_vl'_m \\ \left(-\frac{r}{k+m}\right)wl_ml'_v & wB\left[I_m+\left(-\frac{1}{k+m}\right)l_ml'_m\right]\end{pmatrix}$$

with rank $F=v+m-1$

By imposing the restriction,

$$r=\sum_{i=1}^{v}a_i+B\sum_{i'=v+1}^{v+m}a_i'=0, \tag{3.3}$$

we now solve (3.2) to obtain the LUMV estimator $\hat{\underline{a}}$ of $\underline{a}$. On subtracting the row vectors $\left(-\frac{w}{k+m}\right)\left(rl'_v, Bl'_m\right)$ and $\left(-\frac{1}{k+m}\times\frac{r}{B}\right)\left(rl'_v, Bl'_m\right)$, respectively, from each of the last m row vectors of F and from each of the first v vectors of F, we obtain

$$\underline{F}^*\underline{t}=\underline{P},$$

where

$$\underline{F}^*=\begin{pmatrix}\frac{w(rm+\lambda v)+(r-\lambda)w'}{k+m}I_v+\frac{w'-w}{B(k+m)}(\lambda B-r^2)l_vl'_v, & \underline{0} \\ \underline{0} & wBI_m\end{pmatrix}$$

Now

$$(\underline{F}^*)^{-1}=\begin{pmatrix}\frac{k+m}{w(rm+\lambda v)+w'(r-\lambda)}\left[I_v+\frac{(w'-w)(r^2-\lambda B)}{wBr(m+k)}l_vl'_v\right], & \underline{0} \\ \underline{0} & \frac{1}{wB}I_m\end{pmatrix}$$

so, for $\quad 1\leqq i\leqq v$,

$$\hat{t}_i=\frac{k+m}{w(rm+\lambda v)+(r-\lambda)w'}\left[F_i+\frac{(w'-w)(r^2-\lambda B)}{wBr(m+k)}\sum_{i=1}^{v}F_i\right]$$

$$=\frac{k+m}{(rm+\lambda v)+(r-\lambda)\frac{w'}{w}}\left[\left(Q_i+\frac{w'}{w}Q'_i\right)+\frac{(\lambda B-r^2)}{Br(m+k)}\left(1-\frac{w'}{w}\right)\sum_{i=1}^{v}\left(Q_i+\frac{w'}{w}Q'_i\right)\right] \tag{3.4}$$

and, for $v+1\leqq i\leqq v+m$,

$$\hat{t}_i=\frac{1}{wB}F_i=\frac{1}{wB}wQ_i=(\bar{T}_{i\cdot}-\bar{T}_{\cdot\cdot}),$$

where

$$\bar{T}_{i\cdot}=\frac{1}{r_i}\sum_{j=1}^{B}n_{ij}\,y_{ij},\qquad \bar{T}_{\cdot\cdot}=\frac{1}{v+m}\sum_{i=1}^{v+m}T_i$$

The variance and covariance matrix of $\hat{\underline{a}}$ is given by

$$\begin{pmatrix} \frac{k+m}{w(rm+\lambda v)+w'(r-\lambda)}\left\{I_v+A_o l_v l'_v\right\}, & -\frac{1}{wB(m+k)}l_v l'_m \\ -\frac{1}{wB(m+k)}l_m l'_v\ , & \frac{1}{wB}\left[I_m+\left(-\frac{1}{k+m}\right)l_m l'_m\right] \end{pmatrix}$$

$$\sigma^2=\begin{pmatrix} \frac{k+m}{(rm+\lambda v)+\frac{w'}{w}(r-\lambda)}\left\{I_v+A_1 l_v l'_v\right\}, & -\frac{1}{B(m+k)}l_v l'_m \\ -\frac{1}{B(m+k)}l_m l'_v\ , & \frac{1}{B}\left[I_m+\left(-\frac{1}{k+m}l_m l'_m\right)\right] \end{pmatrix}$$

where $A_o=\dfrac{w'm(r^2-\lambda B)-w[r^2(k+2m)-\lambda Bm]}{wBr(m+k)^2}$ and

$$A_1=\frac{\lambda Bm-r^2(k+2m)-\frac{w'}{w}m(r^2-\lambda B)}{Br(m+k)^2}$$

Thus, we have, $1\leqq i,i'\leqq v$

$$v(\hat{a}_i-\hat{a}_{i'})=2\frac{k+m}{(rm+\lambda v)+\frac{w'}{w}(r-\lambda)}\sigma^2 \qquad (3.5)$$

$a+1\leqq i,i'\leqq v+m$

$$v(\hat{a}_i-\hat{a}_{i'})=\frac{2}{B}\sigma^2$$

$1\leqq i\leqq v,\quad v+1\leqq i'\leqq v+m$, or $v+1\leqq i\leqq v+m,\quad 1\leqq i'\leqq v$

$$v(\hat{a}_i-\hat{a}_{i'})=\frac{(k+m)\sigma^2}{(rm+\lambda v)+\frac{w'}{w}(r-\lambda)}\left\{1+\frac{[\lambda Bm-r^2(k+2m)]-\frac{w'}{w}m(\lambda B-r^2)}{Br(m+k)^2}\right\}$$

$$+\left[\frac{1}{B(m+k)}+\frac{1}{B}\right]\sigma^2$$

$$=\frac{(k+m)\sigma^2}{(rm+\lambda v)+\frac{w'}{w}(r-\lambda)}\left\{1+\frac{[\lambda Bm-r^2(k+2m)]-\frac{w'}{w}m(\lambda B-r^2)}{Br(w+k)^2}\right\}$$

$$+\frac{1}{B}\left(1+\frac{1}{(m+k)}\right)\sigma^2 \qquad (3.6)$$

IV. Analysis under Model I

Assuming that both a_i 's and b_j 's are constants, (2.2) is then a fixed model (or Model I). In this section we shall obtain the LUMV estimator of $\underline{a}$ in the sampling theory framework based on this fixed model. Since it can be shown by applying the Gauses-Markov theorem that the LUMV estimators of the a_i 's satisfies

$$C\underline{a}=\underline{Q},$$

where $E\underline{Q}=C\underline{a}$,

we can then obtain the desired estimators by setting $w'=0$, in the previous setion. thus

$$\hat{\underline{a}}=\begin{pmatrix}\frac{k+m}{rm+\lambda v}\left[I_v+\frac{\lambda B-r^2}{Br(m+k)}l_v l_v'\right] & 0\\ 0\quad , & \frac{1}{B}I_m\end{pmatrix}\underline{Q}$$

or $$\hat{a}_i=\frac{k+m}{rm+\lambda v}\left[Q_i+\frac{\lambda B-r^2}{Br(m+k)}\sum_{i'=1}^{v}Q_{i'}\right] \qquad \text{for } 1\leqq i\leqq v \tag{4.1}$$

and $$\hat{a}_i=\bar{T}_{i.}-\bar{T}_{..} \qquad \text{for } v+1\leqq i\leqq v+m$$

The variance and covariance matrix of $\hat{\underline{a}}$ is given by

$$\sigma^2\begin{pmatrix}\frac{k+m}{rm+\lambda v}\left[I_v+\frac{\lambda Bm-r^2(k+2m)}{Br(m+k)}l_v l_v'\right], & -\frac{1}{B(m+k)}l_v l_m'\\ \left[-\frac{1}{B(m+k)}\right]l_m l_v', & \frac{1}{B}\left[I_m+\left(-\frac{1}{k+m}\right)l_m l_m'\right]\end{pmatrix}$$

Moreover,

$$v(\hat{a}_i-\hat{a}_{i'})=\frac{2(k+m)}{(rm+\lambda v)}\sigma^2, \qquad \text{for } 1\leqq i,\ i'\leqq v \tag{4.2}$$

$$v(\hat{a}_i-\hat{a}_{i'})=\frac{2}{B}\sigma^2, \quad \text{for } v+1\leq i,\ i'\leqq v+m$$

$$v(\hat{a}_i-\hat{a}_{i'})=\left\{\frac{k+m}{rm+\lambda v}\left[1+\frac{\lambda Bm-r^2(k+2m)}{Br(m+k)^2}\right]+\frac{1}{B}\left(1+\frac{1}{m+k}\right)\right\}\sigma^2$$
$$=\left[\frac{k+m}{rm+\lambda v}\left(1-\frac{1}{v}\right)+\frac{1}{Bk}(1+k)\right]\sigma^2 \tag{4.3}$$

for $1\leqq i\leqq v,\ v+1\leqq i'\leqq v+m$ or $v+1\leqq i\leqq v+m,\ 1\leqq i'\leqq v$.

V. Analysis of Variance and Pairwise Comparison of Treatments

From (3.4), (3.5) and (3.6), we see that the LUMV estimator $\hat{\underline{a}}$ of $\underline{a}$ and $v(\hat{a}_i-\hat{a}_{i'})$, $i\neq i'$, are clear of the nuisance parameter w'/w only if w'/w is known. Thus, only in the cases in which w'/w is known ($w'/w=0$ in the case of Model I), we may use $F_{v-1,df_E}=\frac{SSA_1}{SSE}\Big/\frac{(v-1)}{df_E}$ to test the hypothesis Ho: $a_1=a_2=\ldots.=a_{v+m}$ in Table (5.1), and use the ordinary t-distributions to compare the difference of any two treatments as given in Table (5.2).

In the general case of Model III, no exact methods for testing the hypothesis Ho: $a_1=a_2=\ldots.a_{v+m}$ and for the comparison of any two tratments are available. Under the general Model III the estimates of the a_i 's are usually obtained from the $\hat{a}_i$'s given in (3.4) by replacing w and w' by their unbiased estimates. This process, however, introduces further random errors in the $\hat{a}_i$'s and therefore its efficiency may be very low, especially if the sample size is small.

Summary

In this paper the analysis of the ABIB design is considered. The LUMV estimators of the treatments (unknown fixed constants) were obtained for both Model III and Model I in the sampling theory framework. The results are given respectively is Sections III and IV for Model III and for Model I.

ABIB 設計之統計分析

譚 外 元

1. 本文提供一新設計法稱之爲 ABIB，在本質上此設計相當於一 BIB 設計與一逢機完全區集設計之組合。
2. 本文茲假定第三種模型（混合模型）以及第一種模型(固定模型)；在此二種模型之假定下，本文特求出處理效果（未知常數）之線性最佳不偏估值，如（3,4）與（4,1）。
3. 本文之分析係以傳統之樣本理論爲基礎，關於 Bayes 之理論分析基礎將於另文討論之。

Table 5.1

Sources	S. S.	d.f.	Expected value of M.S
Treatments adjusted	$\sum_{i=1}^{v+m} \hat{a}_i F_i = SSA_1$	$v-1$	$\sigma^2 + \frac{\sum_{i=1}^{v+m}\sum_{j=1}^{v+m} f_{ij} a_i a_j}{v-1}$
	$(\sum_{i=1}^{v+m} \hat{a}_i Q_i$ in Model I assumption)		$(\sigma^2 + \frac{\sum_{i=1}^{v+m}\sum_{j=1}^{v+m} C_{ij} a_i a_j}{v-1}$, under Model I assumption)
Blocks unadjusted	$\sum_{j=1}^{B} \frac{B_j^2}{k+m} - \frac{G^2}{B(k+m)} = SSB_2$	$B-1$	$\sigma^2 + (k+m)\sigma_B^2 + \frac{\sum_{i=1}^{v+m} r_i (a_i - \ddot{a})^2 - \sum_{i=1}^{v+m}\sum_{j=1}^{v+m} f_{ij} a_i a_j}{B-1}$
			$(\sum_{i=1}^{v+m}\sum_{j=1}^{v+m} C_{ij} a_i a_j$ under Model I assumption)
Treatments unadjusted	$\sum_{i=1}^{v+m} \frac{T_i^2}{r_i} - \frac{G^2}{B(k+m)} = SSA_2$	$v-1$	$\sigma^2 + \frac{v-k}{v-1}\sigma_B^2 + \frac{\sum_{i=1}^{v+m} r_i (a_i - \ddot{a})^2}{v-1}$
Blocks, adjusted	$SST - SSA_2 - SSE = SSB_1$	$B-1$	$\sigma^2 + \frac{B(k+m)-(v+m)}{B-1}\sigma_B^2$
Error (Intra-block)	$SSE = SST - SSA_1 - SSB_2$ $= SST - SSA_2 - SSB_1$	$B(k+m)-B-v+1$	σ^2
Total	$SST = \sum_{i=1}^{B(k+m)} (Y_i - \bar{Y}_{..})^2$	$B(k+m)-1$	$\sigma^2 + \frac{(k+m)(B-1)}{B(k+m)-1}\sigma_B^2 + \frac{\sum_{i=1}^{v+m} r_i (a_i - \ddot{a})^2}{B(k+m)-1}$

$MSB_1 = SSS_1/(B-1)$, $MSE = SSE/df_E$. So the estimate of w'/w is given by $\frac{v(r-1)+m(B-1)}{(k+m)(B-1)MSB_1/MSE-(v-k)}$

Table 5.1

	Under Model I assumption	Under Model III assumption with w'/w known
$1 \leq i,\ i' \leq v$	$t=\dfrac{a_i=a_i'}{\left[\dfrac{2(k+m)}{rm+\lambda v}\sigma^2\right]^{1/2}}$	$t=\dfrac{a_i-a_i'}{\left[2\dfrac{k+m}{(rm+\lambda v)+\dfrac{w'}{w}(r-\lambda)}\hat{\sigma}^2\right]^{1/2}}$
$v+1 \leq i,\ i' \leq v+m$	$t=\dfrac{a_i-a_i'}{\left(\dfrac{2}{\mathrm{B}}\hat{\sigma}^2\right)^{1/2}}$	$t=\dfrac{a_i-a_i'}{\left(\dfrac{2}{\mathrm{B}}\hat{\sigma}^2\right)^{1/2}}$
$1 \leq i \leq v$ and $v+1 \leq i' \leq v+m$, $1 \leq i' \leq v$ and $v+1 \leq i \leq v+m$	$t=\dfrac{a_i-a_i'}{\left\{\left[\dfrac{k+m}{m+\lambda v}\left(1-\dfrac{1}{v}\right)+\dfrac{1}{\mathrm{B}k}(1+k)\hat{\sigma}^2\right]\right\}^{1/2}}$	$t=\dfrac{a_i-a_i'}{\left\{\dfrac{(k+m)\hat{\sigma}^2}{(rm+\lambda v)+\dfrac{w'}{w}(r-\lambda)}\left[1+(\lambda \mathrm{B}m-r^2\,\overline{k+2m})-\dfrac{w'}{w}m\,\overline{\lambda \mathrm{B}-r^2}\right]+\dfrac{1}{\mathrm{B}}\left(1+\dfrac{1}{m+k}\right)\hat{\sigma}^2\right\}^{1/2}}$

$\hat{\sigma}^2=\mathrm{MS_E}=\dfrac{\mathrm{SS_E}}{df_\mathrm{E}}$