ANALYSIS OF COVARIANCE IN MIXED MODELS

W. Y. TAN[1]

(Received June 25 2967)

1. Introduction

While a random effect model has been considered only by Cochran (1964), a fixed model is usually adopted in the analysis of covariances. As far as the author is aware, it seems that no one has ever attempted to tackle the problem of analysis of covariances in mixed models. This paper is therefore devoted to such a problem with the purpose of developing a test procedure for the treatment effects in the presence of concomitant variabes in mixed models. In this paper we treat the problem from the traditional sampling theory approach; the results from the Bayesian approach will appear latter on somewhere else.

2. Elimination of the Concomitant Variables in Mixed Models

In this paper we consider a mixed model with one concomitant variable X_{ij} as

$$Y_{ij} = \mu^* + \alpha_i^* + \beta_j + X_{ij}\delta + \varepsilon_{ij} = \mu + \alpha_i + \beta_j + x_{ij}\delta + \varepsilon_{ij} \tag{2.1}$$

$$i = 1, 2, \ldots k;\ j = 1, 2, \ldots n.$$

where Y_{ij} are the observations; μ^* is the population mean; α_j^* and δ are the unknown parameters; β_j and ε_{ij} are the random variables;

$$\mu = \mu^* + \bar{X}_{..}\delta + \bar{\alpha}^*,\ \alpha_i = \alpha_i^* - \bar{\alpha}^*,\ x_{ij} = X_{ij} - \bar{X}_{..} \text{ so that } \sum_i \alpha_i = \sum_i \sum_j x_{ij} = 0.$$

Or, in matrix notation,

$$\underline{Y} = 1_{nk}\mu + 1_n \otimes I_k\underline{\alpha} + I_n \otimes 1_k\underline{\beta} + \underline{x}\delta + \underline{\varepsilon}$$

$$= 1_{nk}\mu + 1_n \otimes I_k\underline{\alpha} + \underline{x}\delta + \underline{e} \tag{2.2}$$

where, $\underline{e} = I_n \otimes 1_k\underline{\beta} + \underline{\varepsilon}$,

1. Research Fellow, The Institute of Botany, Academia Sinica, Nankong, Taipei, Taiwan, China.
2. This paper is dedicated to Dr. H. W. Li, the Director of the Inst. of Botany, Academia Sinica, as a token of his 65th Birthday. Paper No. 68 of the Scientific Journal Series, Institute of Botany, Academia Sinica.

$$\underline{Y}' = (y_{11}, y_{12}, \dots y_{1n}, y_{21}, y_{22}, \dots y_{2n}, \dots y_{k1}, y_{k2}, \dots y_{kn})$$

$$\underline{x}' = (x_{11}, x_{12}, \dots x_{1n}, x_{21}, x_{22}, \dots x_{2n}, \dots x_{k1}, x_{k2}, \dots x_{kn})$$

$$\underline{\varepsilon}' = (\varepsilon_{11}, \varepsilon_{12}, \dots \varepsilon_{1n}, \varepsilon_{21}, \varepsilon_{22}, \dots \varepsilon_{2n}, \dots \varepsilon_{k1}, \varepsilon_{k2}, \dots \varepsilon_{kn})$$

1_p is a $p \times 1$ column vector of 1's, and $\otimes$ denotes the "Direct Product".

Using (2.2) we now proceed to illustrate how to eliminate the effect of δ along some principle of projection. In this paper it will be assumed that $E\underline{\beta}=\underline{0}$, $E\underline{\varepsilon} = \underline{0}$, $V_{\underline{\beta}} = \sigma_\beta^2 I_k$ and $V_\varepsilon = \sigma^2 I_{nk}$ and that $\underline{\beta}$ is independent of $\underline{\varepsilon}$, and whence, $E\underline{e} = 0$ and $V_{\underline{e}} = I_n \otimes 1_k 1_k' \sigma_\beta^2 + I_n \otimes I_k \sigma^2 = \sigma^2 C$.

Putting $A_0 = 1_{nk} = 1_n \otimes 1_k$ and $A_1 = 1_n \otimes I_k$, we may then rewrite model (2.2) as

$$\underline{Y} = A_0\mu + A_1\underline{\alpha} + \underline{x}\delta + \underline{e} = A_0\mu + A_{1\cdot0(c)}\underline{\alpha} + \underline{x}_{\cdot0(c)}\delta + \underline{e}$$
$$= A_0\mu + A_{1\cdot0x}\underline{\alpha} + \underline{x}_{\cdot0(c)}\delta^* + \underline{e} \qquad (2.2)$$

where, $A_{1\cdot0(c)} = A_1 - A_0(A_0'C^{-1}A_0)^{-1}A_0'C^{-1}A_1$,

$$\underline{x}_{\cdot0(c)} = \underline{x} - A_0(A_0'C^{-1}A_0)^{-1}A_0'C^{-1}\underline{x},$$

$$A_{1\cdot0x} = A_{1\cdot0(c)} - \underline{x}_{\cdot0(c)}(\underline{x}'_{\cdot0(c)}C^{-1}\underline{x}_{\cdot0(c)})^{-1}\underline{x}'_{\cdot0(c)}C^{-1}A_{1\cdot0(c)},$$

and $\delta^* = \delta + (\underline{x}'_{\cdot0(c)}C^{-1}\underline{x}_{\cdot0(c)})^{-1}\underline{x}'_{\cdot0(c)}C^{-1}A_{1\cdot0(c)}\underline{\alpha}$

Obviously, $A_0'C^{-1}A_{1\cdot0(c)} = \underline{0}$, $A_0'C^{-1}\underline{x}_{\cdot0(c)} = \underline{0}$, $A_0'C^{-1}A_{1\cdot0x} = \underline{0}$

and $A_{1\cdot0x}'C^{-1}\underline{x}_{\cdot0(c)} = \underline{0}.$

Now,

$$C^{-1} = \left\{\frac{\sigma_\beta^2}{\sigma^2} I_n \otimes 1_k 1_k' + I_n \otimes I_k\right\}^{-1} = I_n \otimes \left\{I_k - \frac{\sigma_\beta^2}{k\sigma_\beta^2 + \sigma^2} 1_k 1_k'\right\}$$

On simplification we obtain then,

$$A_{1\cdot0(c)} = 1_n \otimes (I_k - \frac{1}{k} 1_k 1_k'),$$

$$\underline{x}_{\cdot0(c)} = \underline{x},$$

$$A_{1\cdot0x} = 1_n \otimes (I_k - \frac{1}{k} 1_k 1_k') - \frac{n}{w}\underline{x}(\bar{x}_{1\cdot}, \bar{x}_{2\cdot}, \dots \bar{x}_{k\cdot}),$$

and
$$\delta^* = \delta + \frac{n}{w}\sum_{i=1}^{k} \bar{x}_{i\cdot}\,\alpha_i, \qquad (2.4)$$

where,
$$w = \sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}^2 - \frac{k\sigma_\beta^2}{k\sigma_\beta^2 + \sigma^2}\, k\sum_{j=1}^{n}\bar{x}_{\cdot j}^2.$$

Making use of the principle of the general Aitkin's least square method (see, for example, Tan's notes Vol II, chapter XIV, 1967), the LUMV estimators of μ, δ^* and α in the sampling theory framework can readily be obtained as

$$\hat{\mu} = (A_0'C^{-1}A_0)^{-1}A_0'C^{-1}\underline{Y} = \bar{y}_{..},$$

$$\delta^* = (\underline{x}'C^{-1}\underline{x})^{-1}\underline{x}'C^{-1}\underline{Y} = \frac{\sum_i\sum_j x_{ij}y_{ij} - \frac{k\sigma_\beta^2}{k\sigma_\beta^2+\sigma^2}k\sum_j \bar{x}_{.j}\bar{y}_{.j}}{\sum_i\sum_j x_{ij}^2 - \frac{k\sigma_\beta^2}{k\sigma_\beta^2+\sigma^2}k\sum_j \bar{x}_{.j}^2} = \frac{w'}{w},$$

and $$\hat{\underline{\alpha}} = (A_{1\cdot0x}'C^{-1}A_{1\cdot0x} + B'B)^{-1}A_{1\cdot0x}'C^{-1}\underline{Y} \tag{2.5}$$

respectively, where $B = \sqrt{\frac{n}{k}}\,1_k'$.

Since $$A_{1\cdot x0}'C^{-}A_{1\cdot0x} = n(I_k - \frac{1}{k}1_k1_k') - \frac{n^2}{w}\underline{X}_{,.}\underline{X}_{,.}',$$

$$(A_{1\cdot0x}'C^{-1}A_{1\cdot0x} + B'B)^{-1} = \frac{1}{n}\left\{I_k - \frac{n}{w}\underline{X}_{,.}\underline{X}_{,.}'\right\}^{-1}$$

$$= \frac{1}{n}\left\{I_k + \frac{n}{w - n\sum \bar{x}_{i.}^2}\underline{X}_{,.}\underline{X}_{,.}'\right\}$$

and $A_{1\cdot0x}'C^{-1}\underline{Y} = n\left\{(\underline{\bar{Y}}_{,.} - \bar{y}_{..}1_k) - \frac{w'}{w}\underline{X}_{,.}\right\}$, where $\underline{X}_{,.}' = (\bar{x}_{1.}, \bar{x}_{2.}, \ldots \bar{x}_{k.})$

and $\underline{\bar{Y}}_{,.}' = (\bar{y}_{1.}, \bar{y}_{2.}, \ldots \bar{y}_{k.})$, we have then

$$\hat{\underline{\alpha}} = (A_{1\cdot0x}'C^{-1}A_{1\cdot0x} + B'B)^{-1}A_{1\cdot0x}'C^{-1}\underline{Y}$$

$$= (\underline{\bar{Y}}_{,.} - \bar{y}_{..}1_k) - \frac{w' - n\sum_i \bar{x}_{i.}\bar{y}_{i.}}{w - n\sum_i \bar{x}_{i.}^2}\underline{X}_{,.}. \tag{2.6}$$

or

$$\hat{\alpha}_i = (\bar{y}_{i.} - \bar{y}_{..}) - \frac{w' - n\sum_i \bar{x}_{i.}\bar{y}_{i.}}{w - n\sum \bar{x}_{i.}^2}\bar{x}_{i,}. \quad i = 1, 2, \ldots k.$$

The variances of $\hat{\mu}$ and δ^* and the covariance matrix of $\hat{\alpha}$ are obtained respectively as:

$$\mathrm{Var}(\hat{\mu}) = (A_0'C^{-1}A_0)^{-1}A_0'C^{-1}V_{\underline{y}}C^{-1}A_0(A_0'C^{-1}A_0)^{-1}$$

$$= \sigma^2(A_0'C^{-1}A_0)^{-1} = \frac{1}{nk}(k\sigma_\beta^2 + \sigma^2),$$

$$\mathrm{Var}(\delta^*) = (\underline{x}'C^{-1}\underline{x})^{-1}\underline{x}'C^{-1}V_{\underline{y}}C^{-1}\underline{x}(\underline{x}'C^{-1}\underline{x})^{-1} = \sigma^2(\underline{x}'C^{-1}\underline{x})^{-1} = \frac{\sigma^2}{w}$$

and

$$V_{\hat{\underline{\alpha}}} = (A_{1\cdot0x}'C^{-1}A_{1\cdot0x} + B'B)^{-1}A_{1\cdot0x}'C^{-1}V_{\underline{y}}C^{-1}A_{1\cdot0x}(A_{1\cdot0x}'C^{-1}A_{1\cdot0x} + B'B)^{-1}$$

$$= \sigma^2(A_{1\cdot0x}'C^{-1}A_{1\cdot0x} + B'B)^{-1}A_{1\cdot0x}'C^{-1}A_{1\cdot0x}(A_{1\cdot0x}'C^{-1}A_{1\cdot0x} + B'B)^{-1}$$

$$= \sigma^2\frac{1}{n}\left\{I_k - \frac{1}{k}1_k1_k'\right\}\left(I_k + \frac{n}{w - n\sum_i \bar{x}_{i.}^2}\underline{X}_{,.}\underline{X}_{,.}'\right)$$

$$= \frac{\sigma^2}{n}\left\{I_k - \frac{1}{n}1_k1'_k + \frac{n}{w - n\sum_i \bar{x}^2_{i.}}\underline{\bar{X}}_{.}\underline{\bar{X}}'_{.}\right\},$$

a singular matrix with rank $k-1$.

3. Testing the Hgpothesis H_0; $\alpha = 0$

For the purpose of testing H_0; $\underline{\alpha} = \underline{0}$, we assume that $\underline{\beta}$ and ε are normally distributed so that $\underline{Y} \sim N(A_0\mu + A_1\underline{\alpha} + \underline{x}\delta, V_e) = N(A_0\mu + A_{1\cdot 0x}\underline{\alpha} + \underline{x}\delta^*, \sigma^2 C)$. Defining now

$$Q_0(\mu) = (\hat{\mu} - \mu)(A_0'C^{-1}A_0)(\hat{\mu} - \mu) = \frac{nk\sigma^2}{\sigma^2 + k\sigma^2_\beta}(\bar{y}_{..} - \mu)^2,$$

$$Q_\delta(\delta^*) = (\delta^* - \hat{\delta}^*)(\underline{x}'C^{-1}\underline{x})(\delta^* - \hat{\delta}^*) = (\delta^* - \hat{\delta}^*)^2 w,$$

$$Q_\alpha(\underline{\alpha}) = (\underline{\alpha} - \underline{\hat{\alpha}})'(A'_{1\cdot 0x}C^{-1}A_{1\cdot 0x})(\underline{\alpha} - \underline{\hat{\alpha}})$$

$$= n\{\sum_i (\hat{\alpha}_i - \alpha_i)^2 - \frac{n}{w}[\sum_i \bar{x}_{i.}(\hat{\alpha}_i - \alpha_i)]^2\},$$

$$Q_\beta = k\sum_j (\bar{y}_{.j} - \bar{y}_{..})^2 \quad \text{and}$$

$$Q_e = \left\{\sum_i\sum_j (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2 - \frac{\left[\sum_i\sum_j (x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.}) - \frac{k\sigma^2_\beta}{\sigma^2 + k\sigma^2_\beta}k\sum_j \bar{x}_{.j}\bar{y}_{.j}\right]^2}{\left[\sum_i\sum_j (x_{ij} - \bar{x}_{i.})^2 - \frac{k\sigma^2_\beta}{\sigma^2 + k\sigma^2_\beta}k\sum_j \bar{x}^2_{.j}\right]}\right\},$$

it is straightforward to show that

$$(\underline{Y} - A_0\mu - A_{1\cdot 0x}\underline{\alpha} - x\delta^*)'C^{-1}(\underline{Y} - A_0\mu - A_{1\cdot 0x}\underline{\alpha} - x\delta^*)$$

$$= \left\{Q_e + \frac{\sigma^2}{k\sigma^2_\beta + \sigma^2}Q_\beta + Q_0(\mu) + Q_\delta(\delta^*) + Q_\alpha(\underline{\alpha})\right\}$$

Hence, the likelihood function of $(\mu, \delta^*, \underline{\alpha}, \sigma^2, \sigma^2_\beta)$ is

$$L(\mu, \delta^*, \alpha, \sigma^2, \sigma^2_\beta | \underline{y}) = (2\pi)^{-\frac{n\kappa}{2}}(\sigma^2)^{-\frac{n(\kappa-1)}{2}}(\sigma^2 + k\sigma^2_\beta)^{-\frac{n}{2}} \times$$

$$\exp\left\{-\left[\frac{Q_e}{\sigma^2} + \frac{Q_\beta}{k\sigma^2_\beta + \sigma^2} + \frac{1}{\sigma^2}(Q_0(\mu) + Q_\delta(\delta^*) + Q_\alpha(\underline{\alpha}))\right]\right\}$$

Making use of the Cochan-James theorem (see Tan's Notes, Vol II, pp 448, 1967), it can be shown that

$\frac{Q_e}{\sigma^2}$, $\frac{Q_\beta}{k\sigma^2_\beta + \sigma^2}$, $Q_0(\mu)$, $Q_\delta(\delta^*)$ and $Q_\alpha(\alpha)$ are independent Chi-squares with degrees of freedom given by $(k-1)(n-1)-1$, $n-1$, 1, 1 and $k-1$, respectively.

It follows that, under $H_0;\ \alpha = 0$,

$\frac{Q_\alpha(0)}{k-1} \Big/ \frac{Q_e}{(k-1)(n-1)-1} \sim F_{k-1,\ (k-1)(n-1)-1}$, while under the negation of H_0,

$\frac{Q_\alpha(0)}{k-1} \Big/ \frac{Q_e}{(k-1)(n-1)-1} \sim F_{k-1,\ (n-1)(n-1),\ \delta^2}$, a noncentral F with $d.f. = (k-1, (k-1)(n-1)-1)$ and noncentrality parameter

$\delta^2 = \frac{n}{\sigma^2}\left\{\sum \alpha_i^2 - \frac{n}{w}(\sum_i \bar{x}_{i\cdot}\alpha_i)^2\right\}$. Upon simplification we have in fact

$$Q_\alpha(0) = n\sum_i(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 - \frac{w' - n\sum_i \bar{x}_{i\cdot}\bar{y}_{i\cdot}}{w - n\sum \bar{x}_{i\cdot}^2} n\sum \bar{x}_{i\cdot}\bar{y}_{i\cdot}$$

$$- \frac{w'}{w}\left[n\sum_i \bar{x}_{i\cdot}\bar{y}_{i\cdot} - \frac{w' - n\sum_i \bar{x}_{i\cdot}\bar{y}_{i\cdot}}{w - n\sum_i \bar{x}_{i\cdot}^2} n\sum \bar{x}_{i\cdot}^2\right]$$

and
$$Q_\alpha(0) + Q_\delta(0) = n\sum_i(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 + \frac{(w' - n\sum_i \bar{x}_{i\cdot}\bar{y}_{i\cdot})^2}{w - n\sum_i \bar{x}_{i\cdot}^2}.$$

Since w' and w are functions of $\theta = \frac{\sigma_\beta^2}{\sigma^2}$, the statistic $\frac{Q_\alpha(0)}{k-1} \Big/ \frac{Q_e}{(k-1)(n-1)-1}$ is in general a function of $\theta = \frac{\sigma_\beta^2}{\sigma^2}$. Thus in testing the Hypothesis $H_0;\ \alpha = 0$ we should distinguish two cases:

(a) Case 1. If $\theta = \frac{\sigma_\beta^2}{\sigma^2}$ is known, then both $Q_\alpha(0)$ and Q_e are clear of σ_β^2 and σ^2 and further, $\frac{Q_\beta}{(1+k\theta)} \sim \sigma^2\chi^2_{n-1}$, independent of $Q_\alpha(0)$ and Q_e. Hence $Q_e + \frac{Q_\beta}{(1+k\theta)} \sim \sigma^2\chi^2_{k(n-1)-1}$ and the likelihood ratio method yields the critical region at level 0.05 as

$$\text{Co};\ \frac{Q_\alpha(0)}{k-1} \Big/ \frac{Q_e + Q_\beta/(1+k\theta)}{k(n-1)-1} \geqslant F_{k-1,\ k(n-1)-1}(0.05),$$

where $F_{k-1,\ k(n-1)-1}(0.05)$ is the upper 0.05 point of $F_{k-1,\ k(n-1)-1}$.

(b) Case 2. If $\theta = \frac{\sigma_\beta^2}{\sigma^2}$ is unknown, then θ poses as a nuisance parameter in $F_\alpha = \frac{Q_\alpha(0)}{k-1}\ \frac{Q_e}{(k-1)(n-1)-1}$. When n is very large, one may however approximate $Q_\alpha(0)$ and Q_e by substituting for $\theta = \frac{\sigma_\beta^2}{\sigma^2}$ by its unbiased estimate to yield $Q_\alpha(0, \hat{\theta})$ and $Q_e(\hat{\theta})$. The critical region for $H_0;\ \underline{\alpha} = 0$ can be approximated by

$$\text{Co};\ \frac{Q_\alpha(0, \hat{\theta})}{k-1} \Big/ \frac{Q_e}{(k-1)(n-1)-1} \geqslant F_{k-1,\ (k-1)(n-1)-1}(0.05)$$

4. Comments on the Testing Procedure given in 3.

In cases in which $\theta = \frac{\sigma_\beta^2}{\sigma^2}$ is unknown, the procedure suggested in case 2 of section 3 may invalidate the F test when n is small. This is to be expected since the substitution of $\hat{\theta}$ for θ introduces further random errors in $Q_\alpha(0)$ and Q_e. Therefore in cases in which n is small the efficiency of the procedure suggested in case 2 of the previous section may be very low. A number of special cases deserve mention, however.

Case 1. If it is known that $\sigma_\beta^2 = 0$, then Q_β is an unbiased estimate of $(n-1)\sigma^2$. Hence

$$F_1 = \frac{Q_\alpha(0, \sigma_\beta^2 = 0)}{k-1} \Big/ \frac{Q_e(\sigma_\beta^2 = 0) + Q_\beta}{k(n-1)-1} \sim F_{k-1,\, k(n-1)-1},$$

where

$$Q_\alpha(0, \sigma_\beta^2 = 0) = n\sum_i (\bar{y}_{i.} - \bar{y}_{..})^2 - \frac{\sum_i\sum_j (x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.})}{\sum\sum (x_{ij} - \bar{x}_{i.}^2)^2}\, n\sum_i \bar{x}_{i.}\bar{y}_{i.}$$

$$- \frac{\sum_i\sum_j x_{ij}y_{ij}}{\sum_i\sum_j x_{ij}^2}\left[n\sum_i \bar{x}_{i.}\bar{y}_{i.} - \frac{\sum_i\sum_j (x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.})}{\sum_i\sum_j (x_{ij} - \bar{x}_{i.})^2}\, n\sum \bar{x}_{i.}^2\right],$$

$$Q_e(\sigma_\beta^2 = 0) = \left\{\sum_i\sum_j (y_{ij} - \bar{y}_{i.} - \bar{y}_{j} + \bar{y}_{..})^2 - \frac{[\sum_i\sum_j (x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.})]^2}{\sum_i\sum_j (x_{ij} - \bar{x}_{i.})^2}\right\}.$$

Thus, the critical region at level 0.05 for F_0; $\underline{\alpha} = 0$ is
Co: $F_1 \geqslant F_{k-1,\, k(n-1)-1}(0.05)$, where $F_{k-1,\, k(n-1)-1}(0.05)$ is the upper 0.05 point of $F_{k-1,\, k(n-1)-1}$.

Case 2. If $k\sigma_\beta^2 \gg \sigma^2$ so that $\phi = \frac{k\sigma_\beta^2}{k\sigma_\beta^2 + \sigma^2} \sim 1$, then $Q_\alpha(0)$ and Q_e are well approximated by

$$Q_\alpha(0, \phi = 1) = \sum_i (\bar{y}_{i.} - \bar{y}_{..})^2 - \frac{\sum_i\sum_j (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j})(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j})}{\sum_i\sum_j (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j})^2}\, n\sum \bar{x}_{i.}\bar{y}_{i.}$$

$$- \frac{\sum_i\sum_j (x_{ij} - \bar{x}_{.j})(y_{ij} - \bar{y}_{.j})}{\sum_i\sum_j (x_{ij} - \bar{x}_{.j})^2}\left\{n\sum_i \bar{x}_{i.}\bar{y}_{i.} - \frac{\sum_i\sum_j (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j})(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j})}{\sum_i\sum_j (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j})^2}\, n\sum \bar{x}_{i.}^2\right\}$$

and

$$Q_e(\phi = 1) = \left\{\sum_i\sum_j (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2 - \frac{[\sum_i\sum_j (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j})(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j})]^2}{\sum_i\sum_j (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j})^2}\right\},$$

respectively. Therefore the critical region at level 0.05 for H_0; $\underline{\alpha} = 0$ is approximated by

$$\text{Co:}\ \frac{Q_\alpha(0, \phi = 1)}{k-1} \Big/ \frac{Q_e(\phi = 1)}{(k-1)(n-1)-1} \geqslant F_{k-1, (k-1)(n-1)-1}(0.05)$$

5. Computational Procedures

From sections 3 and 4, it follows that in actually applying the results of this paper we may in fact proceed as follows:

(1) Compute Q_β and $Q_e(\sigma^2_\beta = 0)$. If $\frac{Q_\beta}{n-1} \Big/ \frac{Q_e(\sigma^2_\beta = 0)}{(k-1)(n-1)-1} \geqslant F_{n-1, (k-1)(n-1)-1}(0.05)$, we reject the hypothesis H'_0; $\sigma^2_\beta = 0$ and proceed along (2) or (3) given below; if $\frac{Q_\beta}{n-1} \Big/ \frac{Q_e(\sigma^2_\beta = 0)}{(k-1)(n-1)-1} \leqslant F_{n-1, (k-1)(n-1)-1}(0.05)$, we accept H'_0; $\sigma^2_\beta = 0$ and proceed along the procedure given in case 1 of section 4.

(2) Compute $Q_{e1} = \sum_i \sum_j (y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot})^2$. Since $\frac{Q_\beta}{n-1}$ and $\frac{Q_e}{(k-1)(n-1)-1}$ are unbiased for $k\sigma^2_\beta + \sigma^2$ and σ^2 respectively and $Q_{e1} > Q_e$ for all $\theta = \frac{\sigma^2_\beta}{\sigma^2}$, $\frac{Q_\beta}{k-1} \geqslant \frac{Q_{e1}}{(k-1)(n-1)-1}$ would indicate that $\frac{k\sigma^2_\beta}{(k\sigma^2_\beta + \sigma^2)} \sim 1$. When such is the case, we should proceed along the procedure given in case 2 of section 4.

(3) If $\theta = \frac{\sigma^2_\beta}{\sigma^2}$ is known, we should proceed along the procedure given in case a of section 3. If $\theta = \frac{\sigma^2_\beta}{\sigma^2}$ is unknown, if neither (1) nor (2) applies, and if n is very big, we should first obtain estimate $\hat{\theta}$ of $\theta = \frac{\sigma^2_\beta}{\sigma^2}$ and then substitute $\hat{\theta}$ for θ in $Q_\alpha(0)$ and proceed as suggested in case b of section 3. The estimated values of σ^2_β and σ^2 (and hence $\theta = \frac{\sigma^2_\beta}{\sigma^2}$) can be obtained from the following two equations using some numerical method or IBM program.

$$\text{(a)}\quad \sigma^2 = \frac{1}{(k-1)(n-1)-1}\left\{\sum_i \sum_j (y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} + \bar{y}_{\cdot\cdot})^2 - \frac{\left[\sum_i \sum_j (x_{ij} - \bar{x}_{i\cdot})(y_{ij} - \bar{y}_{i\cdot}) - \frac{k\sigma^2_\beta}{\sigma^2 + k\sigma^2_\beta} k \sum_j \bar{x}_{\cdot j}\bar{y}_{\cdot j}\right]}{\sum_i \sum_j (x_{ij} - \bar{x}_{i\cdot})^2 - \frac{k\sigma^2_\beta}{\sigma^2 + k\sigma^2_\beta} k \sum_j \bar{x}^2_{\cdot j}}\right\}$$

and

$$\text{(b)}\quad \frac{Q_\beta}{n-1} = \sigma^2 + k\sigma^2_\beta.$$

6. References

COCHRAN, W. G. (1964). "The Analysis of Covariance", Inst. Stat. Mim. Ser. 6, Univ, North Carolina, Chaple Hill. Quoted from R. L. Anderson and T. A. Bancroft. 1952, Statistical Theory in Research, McGrow-Hill, New York.

TAN, W. Y. (1967). Notes on Mathematical Statistics, Vol II, published by the Institute of Botany, Academia Sinica, Nankang, Taipei, Taiwan, China.