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# APPLICATIONS OF SOME FINITE MARKOV CHAIN THEORIES TO SOME GENETIC PROBLEMS

## Parent-Offspring Mating Type Model with Selection(1)(2)

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### 1. Introduction

This is one of the two papers in which we utilize some finite Markov chain theories to study the effect of selection on the absorption probabilities and the change of frequencies in some inbreeding genetic systems. The present paper is basically concerned with the parent-offspring model as discussed in Horner (1956) and Karlin (1968). The second paper will be concerned with the two locus linkage selfing model as discussed in Nelder (1952) and Karlin (1968). Similar consideration of the application of some finite Markov chain theories to Sib-mating model has also been discussed in Bosso, Sorarrain and Favret (1969).

### 2. Some finite Markov chain results

Consider a finite Markov chain $X(t)$ with stationary transition probabilities and discrete time $t=0, 1, 2....$ In this section we shall derive some simple formulas for the computations of various absorption probabilities, mean absorption times and variances of first absorption times, as well as absolute probability distributions. These results will then be utilized in our models to study the effect of selection on the changes of absorption probabilities and type frequencies.

Let $C_1, C_2, \ldots, C_k$ be the disjoint closed sets of persistent states and $T$ the class of transient states. Then the transition probability matrix (one-step) $P$ can be written in the following canonical form as

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(2) This paper is dedicated to Dr. H. W. Li for his inspiring leadership during his tenure of office as Director of the Institute of Botany, Academia Sinica.

$$P=\begin{bmatrix} P_1 & & & & \\ & P_2 & & \underline{0} & \\ & & \cdot & & \\ \underline{0} & & & \cdot & \\ & & & & \cdot \\ & & & P_k & \\ R_1 & R_2 & \cdots & R_k & Q \end{bmatrix} \tag{2.1}$$

where $P_j$ is the transition matrix corresponding to the closed set $C_j$ and $Q$ the matrix of transition probabilities from transient states to transient states.

From (2.1), we have then

$$P^n=\begin{bmatrix} P_1^n & & & & \\ & P_2^n & & \underline{0} & \\ & & \cdot & & \\ \underline{0} & & & \cdot & \\ & & & & \cdot \\ & & & P_k^n & \\ H_1^{(n)} & H_2^{(n)} & \cdots & H_k^{(n)} & Q^n \end{bmatrix} \tag{2.2}$$

where $H_j^{(1)}=R_j$ and

$H_j^{(n)}=\sum_{s=0}^{n-1} Q^s R_j P_j^{n-1-s}$ is the matrix of absorption probabilities of transient states into states of $C_j$ at and before time $n$.

(i) The absorption probabilities and mean absorption times and variances of first absorption times.

Let $m_j$ be the number of states in $C_j$ and $t$ the number of states in $T$. Furthermore, let $\underset{\sim}{l}_m$ be the $m\times 1$ vector of 1's and put,

$\underset{\sim}{h}_j^{(n)}$ = Vector of absorption probabilities of the transient states into $C_j$ at and before time $n$,

$\underset{\sim}{\rho}_j$ = Vector of ultimate absorption probabilities of the transient states $C_j$,

$\underset{\sim}{\delta}_j^{(n)}$ = Vector of first absorption probabilities of the transient states into $C_j$ at time $n$,

$\underset{\sim}{\delta}^{(n)}=\sum_{j=1}^{k}\underset{\sim}{\delta}_j^{(n)}$ = Vector of first absorption probabilities of the transient states into persistent states at time $n$,

$\underset{\sim}{U}$ = Vector of mean absorption times of the transient states into persistent states,

$\underset{\sim}{V}$ =Vector of variances of first absorption times of the transient states into persistent states,

$\underset{\sim}{\theta}(s)$=Vector of the probability generating functions of the first absorption times of the transient states into persistent states.

Then, since $P_j^m \underset{\sim}{l}_{m_j} = \underset{\sim}{l}_{m_j}$ for all integer $m>0$ and since the absolute value of the eigenvalues of $Q$ is less than 1 (see Karlin (1966), for example), we have

$$\underset{\sim}{h}_j^{(n)} = H_j^{(n)} \underset{\sim}{l}_{m_j} = (I_t + Q + \cdots + Q^{n-1}) R_j \underset{\sim}{l}_{m_j} = (I_t - Q)^{-1}(I_t - Q^n) R_j \underset{\sim}{l}_{m_j}, \quad (2.3)$$

$$\underset{\sim}{\rho}_j = \lim_{n\to\infty} \underset{\sim}{h}_j^{(n)} = (I_t - Q)^{-1} R_j \underset{\sim}{l}_{m_j}, \quad (2.4)$$

$$\underset{\sim}{\delta}_j^{(n)} = Q^{n-1} R_j l_{m_j}, \quad (2.5)$$

$$\underset{\sim}{\delta}^{(n)} = Q^{n-1} \sum_{j=1}^{k} R_j \underset{\sim}{l}_{m_j} = Q^{n-1}(I_t - Q) \underset{\sim}{l}_t \quad (2.6)$$

$$\underset{\sim}{U} = \sum_{n=1}^{\infty} n \delta^{(n)} = \sum_{n=1}^{\infty} n Q^{n-1} (I_t - Q) \underset{\sim}{l}_t = (I_t - Q)^{-2}(I_t - Q)\underset{\sim}{l}_t = (I_t - Q)^{-1} \underset{\sim}{l}_t, \quad (2.7)$$

$$\begin{aligned}\underset{\sim}{V} &= \sum_{n=1}^{\infty} n^2 Q^{n-1}(I_t - Q)\underset{\sim}{l}_t - \underset{\sim}{U}_{sq} \\ &= \{\sum_{n=1}^{\infty} n(n-1)Q^{n-1} + \sum_{n=1}^{\infty} nQ^{n-1}\}(I_t - Q)\underset{\sim}{l}_j - \underset{\sim}{U}_{sq} \\ &= \{2Q(I_t - Q)^{-1} + I_t\}(I_t - Q)^{-1}\underset{\sim}{l}_t - \underset{\sim}{U}_{sq} \\ &= \{2(I_t - Q)^{-1} - I_t\}\underset{\sim}{U} - \underset{\sim}{U}_{sq} \end{aligned} \quad (2.8)$$

where $U'_{sq} = (U_1^2, U_2^2, \ldots, U_t^2)$ with $U' = (U_1, U_2, \ldots, U_t)$, and

$$\underset{\sim}{\theta}(s) = \sum_{n=1}^{\infty} s^n Q^{n-1}(I_t - Q)\underset{\sim}{l}_t = s(I_t - sQ)^{-1}(I_t - Q)\underset{\sim}{l}_t \quad (2.9)$$

(ii) Computations using spectral expansion of $Q$.

Suppose now $Q$ is diagonable so that, using lemma 1 as given in the appendix,

$$Q^n = \sum_{i=1}^{l} \lambda_i^n A_i \quad \text{and} \quad (I_t - Q)^{-1} = \sum_{i=1}^{l} \frac{1}{1-\lambda_i} A_i,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_l$ are the distinct eigenvalues of $Q$ and

$$A_i = \prod_{\substack{j \\ j\neq i}} \frac{1}{(\lambda_i - \lambda_j)} (Q - \lambda_j I_t), \; i = 1, 2, \ldots, l.$$

Then, (2.3)−(2.9) reduce respectively to

$$\underset{\sim}{h}_j^{(n)} = \sum_{i=1}^{l} \frac{1-\lambda_i^n}{1-\lambda_i} A_i R_j \underset{\sim}{l}_{m_j} \quad (2.10)$$

$$\underset{\sim}{\rho}_j = \sum_{i=1}^{l} \frac{1}{1-\lambda_i} A_i R_j \underset{\sim}{l}_{m_j} \quad (2.11)$$

$$\underset{\sim}{\delta}_j^{(n)} = \sum_{i=1}^{l} \lambda_i^{n-1} A_i R_j \underset{\sim}{l}_{m_j} \quad (2.12)$$

$$\underset{\sim}{\delta}^{(n)}=\sum_{i=1}^{l}(\lambda_i^{n-1}-\lambda_i^n)A_i\underset{\sim}{l}_t \tag{2.13}$$

$$\underset{\sim}{U}=\sum_{i=1}^{l}\frac{1}{1-\lambda_i}A_i\underset{\sim}{l}_t \tag{2.14}$$

$$\underset{\sim}{V}=\sum_{i=1}^{l}\frac{1+\lambda_i}{(1-\lambda_i)^2}A_i\underset{\sim}{l}_t-\underset{\sim}{U}_{sq} \tag{2.15}$$

and

$$\underset{\sim}{\theta}(s)=\sum_{i=1}^{l}\frac{s(1-\lambda_i)}{(1-s\lambda_i)}A_i\underset{\sim}{l}_t \tag{2.16}$$

(iii) The absolute probability distributions of $X(n)$ given that $P_j$ and $Q$ are diagonable.

Given the initial distribution of $X(0)$ as $\underset{\sim}{a}_0$, the absolute probability distribution $\underset{\sim}{a}_n$ of $X(n)$ is then

$$\underset{\sim}{a}'_n=\underset{\sim}{a}'_0P^n.$$

If $Q$ and $P_j$ are diagonable so that $Q^n=\sum_{i=1}^{l}\lambda_i^nA_i$ as given before and $P_j^n=\sum_{s=1}^{l_j}\mu_{sj}^nB_{sj}$, where $\mu_{1j}, \mu_{2j}, \ldots, \mu_{l_jj}$ are the distinct eigenvalues of $P_j$ and

$$B_{sj}=\prod_{\substack{s'\\ s'\neq s}}\frac{1}{(\mu_{sj}-\mu_{s'j})}(P_j-\mu_{s'j}I_{m_j}),\ j=1,2,\ldots k,\ \text{then}$$

$$H_j^{(n)}=\sum_{r=0}^{n-1}Q^rR_jP_j^{n-1-r}=\sum_{s=1}^{l}\sum_{t=1}^{l_j}\left(\sum_{r=0}^{n-1}\lambda_s^r\mu_{tj}^{n-1-r}\right)A_sR_jB_{tj}. \tag{2.17}$$

If the $P_j$'s are absorbing classes, then (2.17) reduces to

$$H_j^{(n)}=\sum_{r=0}^{n-1}\sum_{i=1}^{l}\lambda_i^rA_iR_j=\sum_{i=1}^{l}\frac{1-\lambda_i^n}{1-\lambda_i}A_iR_j=\underset{\sim}{P}_j-\sum_{i=1}^{l}\frac{\lambda_i^n}{1-\lambda_i}A_iR_j \tag{2.18}$$

In this case we have then

$$\underset{\sim}{a}'_n=\underset{\sim}{a}'_0\begin{bmatrix}1 & & & & \\ & \cdot & & & \\ & & \cdot & & 0\\ 0 & & & \cdot & \\ & & & 1 & \\ \underset{\sim}{\rho}_1 & \underset{\sim}{\rho}_2 & \cdots & \underset{\sim}{\rho}_k & 0\end{bmatrix}-\sum_{i=1}^{l}\lambda_i^n\underset{\sim}{a}'_0\begin{bmatrix} & 0 & & 0\\ \frac{1}{1-\lambda_i}A_iR_1, & \frac{1}{1-\lambda_i}A_iR_2, & \ldots, \frac{1}{1-\lambda_i}A_iR_k, & A_i\end{bmatrix}$$

which converges to $\underset{\sim}{a}'_0\begin{bmatrix}1 & & & \\ & 1 \cdot & & 0\\ 0 & & \cdot 1 & \\ \underset{\sim}{\rho}_1 & \underset{\sim}{\rho}_2 & \cdots \underset{\sim}{\rho}_k & 0\end{bmatrix}$ as $n\to\infty$.

It is easily seen that this limit is in fact the long-run as well as the stationary distributions of the chain.

### 3. The parent-offspring mating type model with selection

(a) The model.

We consider a single locus with two alleles A:a and the mating is such that the child mates always with the younger parent. Then there are 9 states in the state space and the states are the frequencies of the mating types

| Parent × Offspring |
|---|
| AA × AA |
| aa × aa |
| AA × Aa |
| aa × Aa |
| Aa × AA |
| Aa × aa |
| Aa × Aa |
| aa × AA |
| AA × aa |

To allow for the consideration of the effect of selection, it is assumed in this preliminary report that there is no difference in effects of selection regarding sex; and furthermore, the homozygous genotypes have the same selective values. Then, with no loss of generality, the selective values of the three genotypes can be written as

| | Selective values |
|---|---|
| AA | $x$ |
| Aa | 1 |
| aa | $x$ , |

where $x \geq 0$.

Notice that $x=1$ corresponds to no selection; $x<1$ heterozygote superior and $x>1$, heterozygote inferior. For $x=1$, the above model has been discussed in detail in Horner (1956) and Karlin (1968).

The transition matrix for the above model is given by

| Parent×Offspring | AA×AA | aa×aa | AA×Aa | aa×Aa | Aa×AA | Aa×aa | Aa×Aa | aa×AA | AA×aa |
|---|---|---|---|---|---|---|---|---|---|
| AA×AA | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| aa ×aa | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| AA×Aa | 0 | 0 | 0 | 0 | $\frac{x}{1+x}$ | 0 | $\frac{1}{1+x}$ | 0 | 0 |
| aa ×Aa | 0 | 0 | 0 | 0 | 0 | $\frac{x}{1+x}$ | $\frac{1}{1+x}$ | 0 | 0 |
| Aa ×AA | $\frac{x}{1+x}$ | 0 | $\frac{1}{1+x}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| Aa ×aa | 0 | $\frac{x}{1+x}$ | 0 | $\frac{1}{1+x}$ | 0 | 0 | 0 | 0 | 0 |
| Aa ×Aa | 0 | 0 | 0 | 0 | $\frac{x}{2(1+x)}$ | $\frac{x}{2(1+x)}$ | $\frac{1}{1+x}$ | 0 | 0 |
| aa ×AA | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| AA×aa | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

where $P_1=P_2=1$, $R_1=\begin{pmatrix} 0 \\ 0 \\ \frac{x}{1+x} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $R_2=\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{x}{1+x} \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Q=\begin{pmatrix} 0 & 0 & \frac{x}{1+x} & 0 & \frac{1}{1+x} & 0 & 0 \\ 0 & 0 & 0 & \frac{x}{1+x} & \frac{1}{1+x} & 0 & 0 \\ \frac{1}{1+x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1+x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{x}{2(1+x)} & \frac{x}{2(1+x)} & \frac{1}{1+x} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

The eigenvalues of $Q$ are readily obtained as $\lambda_1=0$, $\lambda_2=-\frac{\sqrt{x}}{1+x}$, $\lambda_3=\frac{\sqrt{x}}{1+x}$, $\lambda_4=\frac{1}{2(1+x)}(1-\sqrt{1+4x})=\frac{\delta_1}{1+x}$ and $\lambda_5=\frac{1}{2(1+x)}(1+\sqrt{1+4x})=\frac{\delta_2}{1+x}$.

Using lemma 2 as given in the appendix and noting that $\sum_{j=1}^{5} A_j=I_7$, we have for the spectral matrices of $Q$:

$$A_1=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2x} & \frac{1}{2x} & 0 & 0 & \frac{1}{x} & 0 & 0 \\ \frac{1}{2x} & \frac{1}{2x} & 0 & 0 & \frac{1}{x} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2}\frac{1+x}{x} & \frac{1}{2}\frac{1+x}{x} & -(1+x) & 0 & \frac{1+x}{x} & 1 & 0 \\ \frac{1}{2}\frac{1+x}{x} & \frac{1}{2}\frac{1+x}{x} & 0 & -(1+x) & \frac{1+x}{x} & 0 & 1 \end{pmatrix}$$

$$A=\frac{1}{4\sqrt{x}}\begin{pmatrix} \sqrt{x} & -\sqrt{x} & -x & x & 0 & 0 & 0 \\ -\sqrt{x} & \sqrt{x} & x & -x & 0 & 0 & 0 \\ -1 & 1 & \sqrt{x} & -\sqrt{x} & 0 & 0 & 0 \\ 1 & -1 & -\sqrt{x} & \sqrt{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(1+x) & (1+x) & \sqrt{x}(1+x) & -\sqrt{x}(1+x) & 0 & 0 & 0 \\ (1+x) & -(1+x) & -\sqrt{x}(1+x) & \sqrt{x}(1+x) & 0 & 0 & 0 \end{pmatrix}$$

$$A_3=\frac{1}{4\sqrt{x}}\begin{pmatrix} \sqrt{x} & -\sqrt{x} & x & -x & 0 & 0 & 0 \\ -\sqrt{x} & \sqrt{x} & -x & x & 0 & 0 & 0 \\ 1 & -1 & \sqrt{x} & -\sqrt{x} & 0 & 0 & 0 \\ -1 & 1 & -\sqrt{x} & \sqrt{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (1+x) & -(1+x) & \sqrt{x}(1+x) & -\sqrt{x}(1+x) & 0 & 0 & 0 \\ -(1+x) & (1+x) & -\sqrt{x}(1+x) & \sqrt{x}(1+x) & 0 & 0 & 0 \end{pmatrix}$$

$$A_4=\frac{1}{\delta_1(\delta_1-\delta_2)}\left(\begin{matrix} \frac{x}{2}\delta_1 & \frac{x}{2}\delta_1 & \frac{x}{2}\delta_1^2 & \frac{x}{2}\delta_1^2 & \delta_1^2 & 0 & 0 \\ \frac{x}{2}\delta_1 & \frac{x}{2}\delta_1 & \frac{x}{2}\delta_1^2 & \frac{x}{2}\delta_1^2 & \delta_1^2 & 0 & 0 \\ \frac{x}{2} & \frac{x}{2} & \frac{x}{2}\delta_1 & \frac{x}{2}\delta_1 & \delta_1 & 0 & 0 \\ \frac{x}{2} & \frac{x}{2} & \frac{x}{2}\delta_1 & \frac{x}{2}\delta_1 & \delta_1 & 0 & 0 \end{matrix}\right.$$

$$
\begin{pmatrix}
\frac{x}{2}\delta_1 & \frac{x}{2}\delta_1 & \frac{x}{2}\delta_1^2 & \frac{x}{2}\delta_1^2 & \delta_1^2 & 0 & 0 \\
\frac{x}{2}(1+x) & \frac{x}{2}(1+x) & \frac{x}{2}\delta_1(1+x) & \frac{x\delta_1}{2}(1+x) & \delta_1(1+x) & 0 & 0 \\
\frac{x}{2}(1+x) & \frac{x}{2}(1+x) & \frac{x}{2}\delta_1(1+x) & \frac{x\delta_1}{2}(1+x) & \delta_1(1+x) & 0 & 0
\end{pmatrix}
$$

and

$$
A_5=\frac{1}{\delta_2^2(\delta_2-\delta_1)}\begin{pmatrix}
\frac{x}{2}\delta_2 & \frac{x}{2}\delta_2 & \frac{x}{2}\delta_2^2 & \frac{x}{2}\delta_2^2 & \delta_2^2 & 0 & 0 \\
\frac{x}{2}\delta_2 & \frac{x}{2}\delta_2 & \frac{x}{2}\delta_2^2 & \frac{x}{2}\delta_2^2 & \delta_2^2 & 0 & 0 \\
\frac{x}{2} & \frac{x}{2} & \frac{x}{2}\delta_2 & \frac{x}{2}\delta_2 & \delta_2 & 0 & 0 \\
\frac{x}{2} & \frac{x}{2} & \frac{x}{2}\delta_2 & \frac{x}{2}\delta_2 & \delta_2 & 0 & 0 \\
\frac{x}{2}\delta_2 & \frac{x}{2}\delta_2 & \frac{x}{2}\delta_2^2 & \frac{x}{2}\delta_2^2 & \delta_2^2 & 0 & 0 \\
\frac{1}{2}x(1+x) & \frac{1}{2}x(1+x) & \frac{x\delta_2}{2}(1+x) & \frac{x\delta_2}{2}(1+x) & \delta_2(1+x) & 0 & 0 \\
\frac{1}{2}x(1+x) & \frac{1}{2}x(1+x) & \frac{x\delta_2}{2}(1+x) & \frac{x\delta_2}{2}(1+x) & \delta_2(1+x) & 0 & 0
\end{pmatrix}
$$

(b) The absorption probabilities

Using the above results and formulale (2.11), we obtain the respective vectors of ultimate absorption probabilities into AA×AA and aa×aa given the transient states as

$$
\underset{\sim}{\rho}_{(1)}=\begin{pmatrix}
\frac{1}{2}+\frac{1}{2}\,\frac{x^2}{1+x+x^2} \\
\frac{1}{2}-\frac{1}{2}\,\frac{x^2}{1+x+x^2} \\
\frac{1}{2}+\frac{1}{2}\,\frac{x+x^2}{1+x+x^2} \\
\frac{1}{2}-\frac{1}{2}\,\frac{x+x^2}{1+x+x^2} \\
\frac{1}{2} \\
\frac{1}{2}+\frac{1}{2}\,\frac{x^2}{1+x+x^2} \\
\frac{1}{2}-\frac{1}{2}\,\frac{x^2}{1+x+x^2}
\end{pmatrix}
\quad \text{and} \quad
\underset{\sim}{\rho}_{(2)}=\begin{pmatrix}
\frac{1}{2}-\frac{1}{2}\,\frac{x^2}{1+x+x^2} \\
\frac{1}{2}+\frac{1}{2}\,\frac{x^2}{1+x+x^2} \\
\frac{1}{2}-\frac{1}{2}\,\frac{x+x^2}{1+x+x^2} \\
\frac{1}{2}+\frac{1}{2}\,\frac{x+x^2}{1+x+x^2} \\
\frac{1}{2} \\
\frac{1}{2}-\frac{1}{2}\,\frac{x^2}{1+x+x^2} \\
\frac{1}{2}+\frac{1}{2}\,\frac{x^2}{1+x+x^2}
\end{pmatrix}.
$$

It is easily seen that both $\frac{x^2}{1+x+x^2}$ and $\frac{x+x^2}{1+x+x^2}$ are monotonic increasing functions of $x$ for $x\geq 0$, and both $=0$ if $x=0$ (homozygotes lethal) and both $\uparrow 1$ as $x\to\infty$. Hence,

$$\underset{\sim}{\rho}'_{(1)}=\underset{\sim}{\rho}'_{(2)}=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \text{ if } x=0, \text{ and}$$

$$\underset{\sim}{\rho}'_{(1)}\to\left(1, 0, 1, 0, \frac{1}{2}, 1, 0\right) \text{ and}$$

$$\underset{\sim}{\rho}'_{(2)}\to\left(0, 1, 0, 1, \frac{1}{2}, 0, 1\right), \text{ as } x\to\infty.$$

Moreover, the following conclusions are trivial:

(a) The effect of selection on the ultimate absorption into AA×AA is opposite to the effect of selection on the ultimate absorption into aa×aa. This follows from the fact that the ultimate absorption probabilities into AA×AA and into aa×aa sum to 1.

(b) The ultimate absorption probabilities into AA×AA given initially AA×Aa, Aa×AA and aa×AA (or the ultimate absorption probabilities into aa×Aa, Aa×aa and AA×aa) increase with $x$ and hence are greater than those of no selection case ($x=1$) if $x>1$ (homozygotes superior) and less than those of no selection case if $x<1$ (heterozygotes superior).

(c) Given initially Aa×Aa, the ultimate absorption probabilities into AA×AA and aa×aa are equal and are not affected by selection.

Given in Table 1 are the ultimate absorption probabilities into AA×AA ($\underset{\sim}{\rho}_{(1)}$) and into aa×aa ($\underset{\sim}{\rho}_{(2)}$) for $x=0$, 0.5, 1.0, 1.5, 2, 2.5 and 3.0.

**Table 1.** *Ultimate absorption probabilities into* AA×AA ($\rho_{(1)}$) *and into* aa×aa ($\rho_{(2)}$) *for various values of* $x$.

| | | $x=0$ | .5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|---|
| AA×Aa | $\rho_{(1)}$ | 0.5 | 0.5714 | 0.6667 | 0.7368 | 0.7857 | 0.8205 | 0.8462 |
| | $\rho_{(2)}$ | 0.5 | 0.4286 | 0.3333 | 0.2632 | 0.2143 | 0.1795 | 0.1538 |
| aa×Aa | $\rho_{(1)}$ | 0.5 | 0.4286 | 0.3333 | 0.2632 | 0.2143 | 0.1795 | 0.1538 |
| | $\rho_{(2)}$ | 0.5 | 0.5714 | 0.6667 | 0.7368 | 0.7857 | 0.8205 | 0.8462 |
| Aa×AA | $\rho_{(1)}$ | 0.5 | 0.7143 | 0.8333 | 0.8947 | 0.9286 | 0.9487 | 0.9615 |
| | $\rho_{(2)}$ | 0.5 | 0.2857 | 0.1667 | 0.1053 | 0.0714 | 0.0513 | 0.0385 |
| Aa×aa | $\rho_{(1)}$ | 0.5 | 0.2857 | 0.1667 | 0.1053 | 0.0714 | 0.0513 | 0.0385 |
| | $\rho_{(2)}$ | 0.5 | 0.7143 | 0.8333 | 0.8947 | 0.9286 | 0.9487 | 0.9615 |
| Aa×Aa | $\rho_{(1)}$ | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| | $\rho_{(2)}$ | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| aa×AA | $\rho_{(1)}$ | 0.5 | 0.5714 | 0.6667 | 0.7368 | 0.7857 | 0.8205 | 0.8462 |
| | $\rho_{(2)}$ | 0.5 | 0.4286 | 0.3333 | 0.2632 | 0.2143 | 0.1795 | 0.1538 |
| AA×aa | $\rho_{(1)}$ | 0.5 | 0.4286 | 0.3333 | 0.2632 | 0.2143 | 0.1795 | 0.1538 |
| | $\rho_{(2)}$ | 0.5 | 0.5714 | 0.6667 | 0.7363 | 0.7857 | 0.8205 | 0.8462 |

Using (2.10), the vectors of absorption probabilities into AA×AA ($\underset{\sim}{h}_1^{(n)}$) and into aa×aa ($\underset{\sim}{h}_2^{(n)}$) at and before time $n$ are obtained respectively as

$$\underset{\sim}{h}_1^{(n)}=\underset{\sim}{\rho}_{(1)}-\frac{1}{2(1+x)^n}\{\underset{\sim}{\Delta}_n+\underset{\sim}{\gamma}_n\} \text{ and } \underset{\sim}{h}_2^{(n)}=\underset{\sim}{\rho}_{(2)}-\frac{1}{2(1+x)^n}\{\underset{\sim}{\Delta}_n-\underset{\sim}{\gamma}_n\},$$

where $\underset{\sim}{\gamma}'_n=\frac{x^{(n+2)/2}}{1+x+x^2}(\gamma_n\sqrt{x}, -\gamma_n\sqrt{x}, \gamma_{n-1}, -\gamma_{n-1}, 0, (1+x)\gamma_{n-1}, -(1+x)\gamma_{n-1}),$

$$\underset{\sim}{\Delta}'_n=\frac{1}{\Delta_1}(\Delta_{n+2}, \Delta_{n+2}, \Delta_{n+1}, \Delta_{n+1}, \Delta_{n+2}, (1+x)\Delta_{n+1}, (1+x)\Delta_{n+1}),$$

$$\gamma_n=\begin{cases}\sqrt{x} \text{ if } n=0 \text{ and even } >0\\(1+x) \text{ if } n \text{ is odd } >0\end{cases}, \quad \text{and}$$

$$\Delta_n=\delta_1^n-\delta_2^n.$$

To see the effect of selection on the absorption probabilities, we notice that $\Delta_n/\Delta_1\geq 0$ and $\uparrow$ with $x$ for $x\geq 0$ if $n\geq 2$; and $\frac{\gamma_n x^{(n+3)/2}}{1+x+x^2}$, $\frac{\gamma_{n-1}x^{(n+2)/2}}{1+x+x^2}$ and $\frac{\gamma_{n-1}(1+x)x^{(n+2)/2}}{1+x+x^2}$ are all $\uparrow$ with $x$ for $x\geq 0$ and $n\geq 1$, and $\geq 0$ for all $x\geq 0$ and all $n\geq 1$. It follows that for each $n\geq 1$ fixed, the absorption probabilities into AA×AA given initially AA×Aa, Aa×AA and aa×AA (or the absorption probabilities into aa×aa given initially aa×Aa, Aa×aa and AA×aa) increase steadily with $x$ for $x\geq 0$. Moreover, since $\frac{\Delta_{n+2}}{\Delta_1}-\gamma_n\frac{x^{(n+3)/2}}{1+x+x^2}$, $\frac{\Delta_{n+1}}{\Delta_1}-\gamma_{n-1}\frac{x^{(n+2)/2}}{1+x+x^2}$ and $\frac{\Delta_{n+1}}{\Delta_1}-\gamma_{n-1}\frac{(1+x)x^{(n+2)/2}}{1+x+x^2}$ increase first with $x$ and then decrease with $x$ for $x\geq 0$ and for each $n$ fixed, so are the absorption probabilities into aa×aa given initially AA×Aa and aa×AA (or into AA×AA given initially aa×Aa, Aa×aa and AA×aa). For $x=0$, $\frac{\Delta_n}{\Delta_1}=1$ and $\underset{\sim}{\gamma}_n=\underset{\sim}{0}$ for all $n\geq 1$ so that $\underset{\sim}{h}_1^{(n)}=\underset{\sim}{h}_2^{(n)}=\underset{\sim}{0}$ for all $n$. Given in Table 2 are the numerical values of $x=0.5$, 1, 1.5 and $n=1, 2, \ldots, 10$. It is observed that the absorption probabilities increase steadily with $n$ for each $x\geq 0$ fixed as it is expected intuitively.

Using (2.12), we obtain the respective first time absorption vectors at time $n$ into AA×AA ($\underset{\sim}{\delta}_1^{(n)}$) and into aa×aa ($\underset{\sim}{\delta}_2^{(n)}$) as:

$$\underset{\sim}{\delta}_1^{(1)\prime}=(0, 0, \frac{x}{1+x}, 0, 0, 0, 0),$$

$$\underset{\sim}{\delta}_2^{(1)\prime}=(0, 0, 0, \frac{x}{1+x}, 0, 0, 0) \text{ and for } n\geq 2,$$

$$\underset{\sim}{\delta}_1^{(n)}=\frac{x}{4(1+x)^n}\{2x\underset{\sim}{\Delta}_{n-3}+\underset{\sim}{\varepsilon}_n\} \text{ and}$$

$$\underset{\sim}{\delta}_2^{(n)}=\frac{x}{4(1+x)^n}\{2x\underset{\sim}{\Delta}_{n-3}+\underset{\sim}{\varepsilon}_n\}, \text{ where}$$

$$\underset{\sim}{\varepsilon}'_n=x^{\frac{n-1}{2}}(\sqrt{x}\varepsilon_n, -\sqrt{x}\varepsilon_n, \varepsilon_{n-1}, -\varepsilon_{n-1}, 0, (1+x)\varepsilon_{n-1}, -(1+x)\varepsilon_{n-1})$$

with $\varepsilon_n=\begin{cases}2 \text{ if } n=0 \text{ or even } >0\\0 \text{ if } n \text{ odd } >0\end{cases}$

**Table 2.** *Absorption probabilities into* AA×AA ($\underset{\sim}{h}_1^{(n)}$) *and into* aa×aa ($\underset{\sim}{h}_2^{(n)}$) *at and before time n for x=0.5, 1, 1.5.*

| | AA×Aa | aa×Aa | Aa×AA | Aa×aa | Aa×Aa | aa×AA | AA×aa |
|---|---|---|---|---|---|---|---|
| $x=0.5$ | | | | | | | |
| $n=1$ $h_1^{(n)}$ | 0.0 | 0.0 | 0.3333 | 0.0 | 0.0 | 0.0 | 0.0 |
| $h_2^{(n)}$ | 0.0 | 0.0 | 0.0 | 0.3333 | 0.0 | 0.0 | 0.0 |
| $n=2$ $h_1^{(n)}$ | 0.1111 | 0.0 | 0.3333 | 0.0 | 0.0556 | 0.0 | 0.0 |
| $h_2^{(n)}$ | 0.0 | 0.1111 | 0.0 | 0.3333 | 0.0556 | 0.0 | 0.0 |
| $n=3$ $h_1^{(n)}$ | 0.1481 | 0.0370 | 0.4074 | 0.0 | 0.0926 | 0.1111 | 0.0 |
| $h_2^{(n)}$ | 0.0370 | 0.1481 | 0.0 | 0.4074 | 0.0926 | 0.0 | 0.1111 |
| $n=4$ $h_1^{(n)}$ | 0.1975 | 0.0617 | 0.4321 | 0.0247 | 0.1296 | 0.1481 | 0.0370 |
| $h_2^{(n)}$ | 0.0617 | 0.1975 | 0.0247 | 0.4321 | 0.1296 | 0.0370 | 0.1481 |
| $n=5$ $h_1^{(n)}$ | 0.2305 | 0.0947 | 0.4650 | 0.0412 | 0.1626 | 0.1975 | 0.0617 |
| $h_2^{(n)}$ | 0.0947 | 0.2305 | 0.0412 | 0.4650 | 0.1626 | 0.0617 | 0.1975 |
| $n=6$ $h_1^{(n)}$ | 0.2634 | 0.1221 | 0.4870 | 0.0631 | 0.1927 | 0.2305 | 0.0947 |
| $h_2^{(n)}$ | 0.1221 | 0.2634 | 0.0631 | 0.4870 | 0.1927 | 0.9047 | 0.2305 |
| $n=7$ $h_1^{(n)}$ | 0.2908 | 0.1495 | 0.5089 | 0.0814 | 0.2202 | 0.2634 | 0.1221 |
| $h_2^{(n)}$ | 0.1495 | 0.2908 | 0.0814 | 0.5089 | 0.2202 | 0.1221 | 0.2634 |
| $n=8$ $h_1^{(n)}$ | 0.3164 | 0.1739 | 0.5272 | 0.0997 | 0.2452 | 0.2908 | 0.1495 |
| $h_2^{(n)}$ | 0.1739 | 0.3164 | 0.0997 | 0.5272 | 0.2452 | 0.1495 | 0.2908 |
| $n=9$ $h_1^{(n)}$ | 0.3392 | 0.1967 | 0.5443 | 0.1159 | 0.2679 | 0.3164 | 0.1739 |
| $h_2^{(n)}$ | 0.1967 | 0.2392 | 0.1159 | 0.5443 | 0.2679 | 0.1739 | 0.3164 |
| $n=10$ $h_1^{(n)}$ | 0.3600 | 0.2173 | 0.5595 | 0.1311 | 0.2887 | 0.3392 | 0.1967 |
| $h_2^{(n)}$ | 0.2173 | 0.3600 | 0.1311 | 0.5595 | 0.2887 | 0.1967 | 0.3392 |
| $x=1$ | | | | | | | |
| $n=1$ $h_1^{(n)}$ | 0.0 | 0.0 | 0.5 | 0.0 | 0.0 | 0.0 | 0.0 |
| $h_2^{(n)}$ | 0.0 | 0.0 | 0.0 | 0.5 | 0.0 | 0.0 | 0.0 |
| $n=2$ $h_1^{(n)}$ | 0.25 | 0.0 | 0.5 | 0.0 | 0.125 | 0.0 | 0.0 |
| $h_2^{(n)}$ | 0.0 | 0.25 | 0.0 | 0.5 | 0.125 | 0.0 | 0.0 |
| $n=3$ $h_1^{(n)}$ | 0.3125 | 0.0625 | 0.625 | 0.0 | 0.1875 | 0.25 | 0.0 |
| $h_2^{(n)}$ | 0.0625 | 0.3125 | 0.0 | 0.625 | 0.1875 | 0.0 | 0.25 |
| $n=4$ $h_1^{(n)}$ | 0.4063 | 0.0938 | 0.6563 | 0.0313 | 0.25 | 0.3125 | 0.0625 |
| $h_2^{(n)}$ | 0.0938 | 0.4063 | 0.0313 | 0.6563 | 0.25 | 0.0625 | 0.3125 |
| $n=5$ $h_1^{(n)}$ | 0.4531 | 0.1406 | 0.7031 | 0.0469 | 0.2965 | 0.4063 | 0.0938 |
| $h_2^{(n)}$ | 0.1406 | 0.4531 | 0.0469 | 0.7031 | 0.2965 | 0.0938 | 0.4063 |
| $n=6$ $h_1^{(n)}$ | 0.5 | 0.1719 | 0.7266 | 0.0703 | 0.3359 | 0.4531 | 0.1406 |
| $h_2^{(n)}$ | 0.1719 | 0.5 | 0.0703 | 0.7266 | 0.3359 | 0.1406 | 0.4531 |
| $n=7$ $h_1^{(n)}$ | 0.5313 | 0.2031 | 0.75 | 0.0859 | 0.3672 | 0.5 | 0.1719 |
| $h_2^{(n)}$ | 0.2031 | 0.5313 | 0.0859 | 0.75 | 0.3672 | 0.1719 | 0.5 |
| $n=8$ $h_1^{(n)}$ | 0.5586 | 0.2266 | 0.7656 | 0.1016 | 0.3926 | 0.5313 | 0.2031 |
| $h_2^{(n)}$ | 0.2266 | 0.5586 | 0.1016 | 0.7656 | 0.3926 | 0.2031 | 0.5313 |
| $n=9$ $h_1^{(n)}$ | 0.5791 | 0.2471 | 0.7793 | 0.1133 | 0.4131 | 0.5586 | 0.2266 |
| $h_2^{(n)}$ | 0.2471 | 0.5791 | 0.1133 | 0.7793 | 0.4131 | 0.2266 | 0.5586 |

| | | AA×Aa | aa×Aa | Aa×AA | Aa×aa | Aa×Aa | aa×AA | AA×aa |
|---|---|---|---|---|---|---|---|---|
| $n=10$ | $h_1^{(n)}$ | 0.5962 | 0.2632 | 0.7896 | 0.1235 | 0.4297 | 0.5791 | 0.2471 |
| | $h_2^{(n)}$ | 0.2632 | 0.5962 | 0.1235 | 0.7896 | 0.4297 | 0.2471 | 0.5791 |
| $x=1.5$ | | | | | | | | |
| $n=1$ | $h_1^{(n)}$ | 0.0 | 0.0 | 0.6 | 0.0 | 0.0 | 0.0 | 0.0 |
| | $h_2^{(n)}$ | 0.0 | 0.0 | 0.0 | 0.6 | 0.0 | 0.0 | 0.0 |
| $n=2$ | $h_1^{(n)}$ | 0.36 | 0.0 | 0.6 | 0.0 | 0.18 | 0.0 | 0.0 |
| | $h_2^{(n)}$ | 0.0 | 0.36 | 0.0 | 0.6 | 0.18 | 0.0 | 0.0 |
| $n=3$ | $h_1^{(n)}$ | 0.432 | 0.072 | 0.744 | 0.0 | 0.252 | 0.36 | 0.0 |
| | $h_2^{(n)}$ | 0.072 | 0.432 | 0.0 | 0.744 | 0.252 | 0.0 | 0.36 |
| $n=4$ | $h_1^{(n)}$ | 0.5472 | 0.1008 | 0.7728 | 0.0288 | 0.324 | 0.432 | 0.072 |
| | $h_2^{(n)}$ | 0.1008 | 0.5472 | 0.0288 | 0.7728 | 0.324 | 0.072 | 0.432 |
| $n=5$ | $h_1^{(n)}$ | 0.5933 | 0.1469 | 0.8189 | 0.0403 | 0.3701 | 0.5472 | 0.1008 |
| | $h_2^{(n)}$ | 0.1469 | 0.5933 | 0.0403 | 0.8189 | 0.3701 | 0.1008 | 0.5472 |
| $n=6$ | $h_1^{(n)}$ | 0.6394 | 0.1722 | 0.8373 | 0.0588 | 0.4058 | 0.5933 | 0.1469 |
| | $h_2^{(n)}$ | 0.1722 | 0.6394 | 0.0588 | 0.8373 | 0.4058 | 0.1469 | 0.5933 |
| $n=7$ | $h_1^{(n)}$ | 0.6647 | 0.1976 | 0.8557 | 0.0689 | 0.4311 | 0.6394 | 0.1722 |
| | $h_2^{(n)}$ | 0.1976 | 0.6647 | 0.0689 | 0.8557 | 0.4311 | 0.1722 | 0.6394 |
| $n=8$ | $h_1^{(n)}$ | 0.6850 | 0.2138 | 0.8659 | 0.0790 | 0.4498 | 0.6647 | 0.1976 |
| | $h_2^{(n)}$ | 0.2138 | 0.6859 | 0.0790 | 0.8659 | 0.4498 | 0.1976 | 0.6647 |
| $n=9$ | $h_1^{(n)}$ | 0.6995 | 0.2274 | 0.8744 | 0.0855 | 0.4634 | 0.6859 | 0.2138 |
| | $h_2^{(n)}$ | 0.2274 | 0.6995 | 0.0855 | 0.8744 | 0.4634 | 0.2138 | 0.6859 |
| $n=10$ | $h_1^{(n)}$ | 0.71 | 0.2367 | 0.8798 | 0.0909 | 0.4733 | 0.6995 | 0.2274 |
| | $h_2^{(n)}$ | 0.2367 | 0.71 | 0.0909 | 0.8798 | 0.4733 | 0.2274 | 0.6995 |

From above it is obvious that $\underset{\sim}{\delta}_1^{(n)}=\underset{\sim}{\delta}_2^{(n)}=\underset{\sim}{0}$ if $x=0$ while the change of the first time absorption probabilities with respect to $x$ and $n$ is determined by the functions $\varepsilon_n$ and $\Delta_n/\Delta_1$ and by the initially given mating types. For each $n$ fixed, the general picture appears to be that the first time absorption probabilities increase first with $x$ and then decrease as $x$ increases. For each $x$ fixed, the first time absorption probabilities in general progress downward as $n$ increases in a zigzag manner such that it may not decrease in any two adjacent times; it decreases however in each of the subsequences $\{2m\}$, and $\{2m+1\}$, $m=0, 1, 2, \ldots$. Furthermore, from the definition of $\varepsilon_n$, it follows that at any even time the first time absorption probabilities into AA×AA and into aa×aa are equal if given initially the mating types Aa×AA, Aa×aa, Aa×AA, aa×AA and AA×aa while at any odd time, the first time absorption probabilities into the two absorbing classes are equal if given initially the mating types AA×Aa and aa×Aa. Given in table 3 are the respective first time absorption probabilities into AA×AA and aa×aa for $x=0.5$, 1, 1.5 and 2 and for $n=2$, 3, 4, 5, 6, 7, 8.

**Table 3.** *First time absorption probabilities into* AA×AA ($\underset{\sim}{\delta}_1^{(n)}$) *and into* aa×aa ($\underset{\sim}{\delta}_2^{(n)}$) *at time n for x=0.5, 1, 1.5, 2.0 and n=2, 3, 4, 5, 6, 7, 8.*

| $\delta_1^{(n)}$ | AA×Aa | aa×Aa | Aa×AA | Aa×aa | Aa×Aa | aa×AA | AA×aa |
|---|---|---|---|---|---|---|---|
| $n=2$ | | | | | | | |
| $x=0.5$ | 0.1111 | 0.0000 | 0.0000 | 0.0000 | 0.0556 | 0.0000 | 0.0000 |
| $x=1.0$ | 0.2500 | 0.0000 | 0.0000 | 0.0000 | 0.1250 | 0.0000 | 0.0000 |
| $x=1.5$ | 0.3600 | 0.0000 | 0.0000 | 0.0000 | 0.1800 | 0.0000 | 0.0000 |
| $x=2.0$ | 0.4444 | 0.0000 | 0.0000 | 0.0000 | 0.2222 | 0.0000 | 0.0000 |
| $n=3$ | | | | | | | |
| $x=0.5$ | 0.0370 | 0.0370 | 0.0741 | 0.0000 | 0.0370 | 0.1111 | 0.0000 |
| $x=1.0$ | 0.0625 | 0.0625 | 0.1250 | 0.0000 | 0.0625 | 0.2500 | 0.0000 |
| $x=1.5$ | 0.0720 | 0.0720 | 0.1440 | 0.0000 | 0.0720 | 0.3600 | 0.0000 |
| $x=2.0$ | 0.0741 | 0.0741 | 0.1481 | 0.0000 | 0.0741 | 0.4444 | 0.0000 |
| $n=4$ | | | | | | | |
| $x=0.5$ | 0.0494 | 0.0247 | 0.0247 | 0.0247 | 0.0370 | 0.0370 | 0.0370 |
| $x=1.0$ | 0.0937 | 0.0312 | 0.0313 | 0.0313 | 0.0625 | 0.0625 | 0.0625 |
| $x=1.5$ | 0.1152 | 0.0288 | 0.0288 | 0.0288 | 0.0720 | 0.0720 | 0.0720 |
| $x=2.0$ | 0.1235 | 0.0247 | 0.0247 | 0.0247 | 0.0741 | 0.0741 | 0.0741 |
| $n=5$ | | | | | | | |
| $x=0.5$ | 0.0329 | 0.0329 | 0.0329 | 0.0165 | 0.0329 | 0.0494 | 0.0247 |
| $x=1.0$ | 0.0469 | 0.0466 | 0.0469 | 0.0156 | 0.0469 | 0.0937 | 0.0312 |
| $x=1.5$ | 0.0461 | 0.0461 | 0.0461 | 0.0115 | 0.0461 | 0.1152 | 0.0288 |
| $x=2.0$ | 0.0412 | 0.0412 | 0.0412 | 0.0082 | 0.0412 | 0.1235 | 0.0247 |
| $n=6$ | | | | | | | |
| $x=0.5$ | 0.0329 | 0.0274 | 0.0219 | 0.0219 | 0.0302 | 0.0329 | 0.0329 |
| $x=1.0$ | 0.0469 | 0.0312 | 0.0234 | 0.0234 | 0.0391 | 0.0469 | 0.0469 |
| $x=1.5$ | 0.0461 | 0.0253 | 0.0184 | 0.0184 | 0.0357 | 0.0461 | 0.0461 |
| $x=2.0$ | 0.0412 | 0.0192 | 0.0137 | 0.0137 | 0.0302 | 0.0412 | 0.0412 |
| $n=7$ | | | | | | | |
| $x=0.5$ | 0.0274 | 0.0274 | 0.0219 | 0.0183 | 0.0274 | 0.0329 | 0.0274 |
| $x=1.0$ | 0.0312 | 0.0312 | 0.0234 | 0.0156 | 0.0312 | 0.0469 | 0.0312 |
| $x=1.5$ | 0.0253 | 0.0253 | 0.0184 | 0.0101 | 0.0253 | 0.0461 | 0.0253 |
| $x=2.0$ | 0.0192 | 0.0192 | 0.0137 | 0.0064 | 0.0192 | 0.0412 | 0.0192 |
| $n=8$ | | | | | | | |
| $x=0.5$ | 0.0256 | 0.0244 | 0.0183 | 0.0183 | 0.0250 | 0.0274 | 0.0274 |
| $x=1.0$ | 0.0273 | 0.0234 | 0.0156 | 0.0156 | 0.0254 | 0.0312 | 0.0312 |
| $x=1.5$ | 0.0212 | 0.0162 | 0.0101 | 0.0101 | 0.0187 | 0.0253 | 0.0253 |
| $x=2.0$ | 0.0155 | 0.0107 | 0.0064 | 0.0064 | 0.0131 | 0.0192 | 0.0192 |
| $n=2$ | | | | | | | |
| $x=0.5$ | 0.0000 | 0.1111 | 0.0000 | 0.0000 | 0.0556 | 0.0000 | 0 0000 |

| $\delta_2^{(n)}$ | AA×Aa | aa×Aa | Aa×AA | Aa×aa | Aa×Aa | aa×AA | AA×aa |
|---|---|---|---|---|---|---|---|
| $x=1.0$ | 0.0000 | 0.2500 | 0.0000 | 0.0000 | 0.1250 | 0.0000 | 0.0000 |
| $x=1.5$ | 0.0000 | 0.3600 | 0.0000 | 0.0000 | 0.1800 | 0.0000 | 0.0000 |
| $x=2.0$ | 0.0000 | 0.4444 | 0.0000 | 0.0000 | 0.2222 | 0.0000 | 0.0000 |
| $n=3$ | | | | | | | |
| $x=0.5$ | 0.0370 | 0.0370 | 0.0000 | 0.0741 | 0.0370 | 0.0000 | 0.1111 |
| $x=1.0$ | 0.0625 | 0.0625 | 0.0000 | 0.1250 | 0.0625 | 0.0000 | 0.2500 |
| $x=1.5$ | 0.0720 | 0.0720 | 0.0000 | 0.1440 | 0.0720 | 0.0000 | 0.3600 |
| $x=2.0$ | 0.0741 | 0.0741 | 0.0000 | 0.1481 | 0.0741 | 0.0000 | 0.4444 |
| $n=4$ | | | | | | | |
| $x=0.5$ | 0.0247 | 0.0494 | 0.0247 | 0.0247 | 0.0370 | 0.0370 | 0.0370 |
| $x=1.0$ | 0.0312 | 0.0937 | 0.0313 | 0.0313 | 0.0625 | 0.0625 | 0.0625 |
| $x=1.5$ | 0.0288 | 0.1152 | 0.0288 | 0.0288 | 0.0720 | 0.0720 | 0.0720 |
| $x=2.0$ | 0.0247 | 0.1235 | 0.0247 | 0.0247 | 0.0741 | 0.0741 | 0.0741 |
| $n=5$ | | | | | | | |
| $x=0.5$ | 0.0329 | 0.0329 | 0.0165 | 0.0329 | 0.0329 | 0.0247 | 0.0494 |
| $x=1.0$ | 0.0469 | 0.0469 | 0.0156 | 0.0569 | 0.0469 | 0.0312 | 0.0937 |
| $x=1.5$ | 0.0461 | 0.0461 | 0.0115 | 0.0461 | 0.0461 | 0.0288 | 0.1152 |
| $x=2.0$ | 0.0412 | 0.0412 | 0.0082 | 0.0412 | 0.0412 | 0.0247 | 0.1235 |
| $n=6$ | | | | | | | |
| $x=0.5$ | 0.0274 | 0.0329 | 0.0219 | 0.0219 | 0.0302 | 0.0329 | 0.0329 |
| $x=1.0$ | 0.0312 | 0.0469 | 0.0234 | 0.0234 | 0.0391 | 0.0469 | 0.0469 |
| $x=1.5$ | 0.0253 | 0.0461 | 0.0184 | 0.0184 | 0.0357 | 0.0461 | 0.0461 |
| $x=2.0$ | 0.0192 | 0.0412 | 0.0137 | 0.0137 | 0.0302 | 0.0412 | 0.0412 |
| $n=7$ | | | | | | | |
| $x=0.5$ | 0.0274 | 0.0274 | 0.0183 | 0.0219 | 0.0274 | 0.0274 | 0.0329 |
| $x=1.0$ | 0.0312 | 0.0312 | 0.0156 | 0.0234 | 0.0312 | 0.0312 | 0.0469 |
| $x=1.5$ | 0.0253 | 0.0253 | 0.0101 | 0.0184 | 0.0253 | 0.0253 | 0.0461 |
| $x=2.0$ | 0.0192 | 0.0192 | 0.0064 | 0.0137 | 0.0192 | 0.0192 | 0.0412 |
| $n=8$ | | | | | | | |
| $x=0.5$ | 0.0244 | 0.0256 | 0.0183 | 0.0183 | 0.0250 | 0.0274 | 0.0274 |
| $x=1.0$ | 0.0234 | 0.0273 | 0.0156 | 0.0156 | 0.0254 | 0.0312 | 0.0312 |
| $x=1.5$ | 0.0162 | 0.0212 | 0.0101 | 0.0101 | 0.0187 | 0.0293 | 0.0253 |
| $x=2.0$ | 0.0107 | 0.0155 | 0.0064 | 0.0064 | 0.0131 | 0.0192 | 0.0192 |

(c) The mean absorption times $\underset{\sim}{U}$ and the variances $\underset{\sim}{V}$ of first absorption times.

Using (2.14) and (2.15), the vectors of mean absorption times ($\underset{\sim}{U}$) and of the variances of first absorption times ($\underset{\sim}{V}$) are obtained respectively as:

$$\underset{\sim}{U}'=\Big(\frac{1+x}{x}+\Big(\frac{1+x}{x}\Big)^2,\ \frac{1+x}{x}+\Big(\frac{1+x}{x}\Big)^2,\ \Big(\frac{1+x}{x}\Big)^2,\ \Big(\frac{1+x}{x}\Big)^2,\ \frac{1+x}{x}+\Big(\frac{1+x}{x}\Big)^2,\ 1+\frac{1+x}{x}+\Big(\frac{1+x}{x}\Big)^2,\ 1+\frac{1+x}{x}+\Big(\frac{1+x}{x}\Big)^2\Big)$$

and

$$\underset{\sim}{V}' = \frac{1}{x^4}(g_1(x),\ g_1(x),\ g_2(x),\ g_2(x),\ g_1(x),\ g_1(x),\ g_1(x)),$$

where

$$g_1(x) = 1+6x+10x^2+5x^3 \text{ and } g_2(x) = 1+6x+9x^2+4x^3.$$

It is easily seen that $\frac{1+x}{x}$, $\frac{1}{x^4}g_1(x)$ and $\frac{1}{x^4}g_2(x)$ $\downarrow$ as $x$ increases for $x \geq 0$. Hence,

$$\underset{\sim}{U}' \downarrow (2, 2, 1, 1, 2, 3, 3) \text{ and}$$

$\underset{\sim}{V}' \downarrow \underset{\sim}{0}'$ as $x \to \infty$. Furthermore, the means and variances are greater than (less than) those of no selection case if $x<1$ ($x>1$).

It might be of interest to point out that it takes one more generation for aa×AA (or AA×aa) than AA×Aa (or aa×Aa) to reach fixation. This follows from the trivial result that with probability 1 aa×AA→AA×Aa (AA×aa→aa ×Aa) in one generation.

This may also account for the equality of variances of first time absorption probabilities. Given in Table 4 are the mean absorption times and variances of first absorption times for $x=0$, 0.5, 1, 1.5, 2, 2.5, 4.0.

**Table 4.** *Mean absorption times* ($U$) *and variance* ($V$) *of first absorption times*

| $\underset{\sim}{u}$ | $x=0.5$ | 1.0 | 1.5 | 2.0 | 2.5 | 4.0 |
|---|---|---|---|---|---|---|
| AA×Aa | 12 | 6 | 4.4444 | 3.75 | 3.36 | 2.8125 |
| aa×Aa | 12 | 6 | 4.4444 | 4.75 | 3.36 | 2.8125 |
| Aa×AA | 9 | 4 | 2.7778 | 2.25 | 1.96 | 1.5625 |
| Aa×aa | 9 | 4 | 2.7778 | 2.25 | 1.96 | 1.5625 |
| Aa×Aa | 12 | 6 | 4.4444 | 3.75 | 3.36 | 2.8125 |
| aa×AA | 13 | 7 | 5.4444 | 4.75 | 4.36 | 3.8125 |
| AA×aa | 13 | 7 | 5.4444 | 4.75 | 4.36 | 3.8125 |
| $\underset{\sim}{v}$ | $x=0.5$ | 1.0 | 1.5 | 2.0 | 2.5 | 4.0 |
| AA×Aa | 114 | 22 | 9.7531 | 5.8125 | 4.0096 | 1.9727 |
| aa×Aa | 114 | 22 | 9.7531 | 5.8125 | 4.0096 | 1.9727 |
| Aa×AA | 108 | 20 | 8.6420 | 5.0625 | 3.4496 | 1.6602 |
| Aa×aa | 108 | 20 | 8.6420 | 5.0625 | 3.4496 | 1.6602 |
| Aa×Aa | 114 | 22 | 9.7531 | 5.8125 | 4.0096 | 1.9727 |
| aa×AA | 114 | 22 | 9.7531 | 5.8125 | 4.0096 | 1.9727 |
| AA×aa | 114 | 22 | 9.7531 | 5.8125 | 4.0096 | 1.9727 |

(d) The absolute probabilities distributions of $X(n)$.

Given the initial frequencies as $\underset{\sim}{a}_0$, the absolute probability distribution at time $n$ is

$$\underset{\sim}{a}'_n=\underset{\sim}{a}'_0\begin{pmatrix} 1 & 0 & \vdots & \underset{\sim}{0} \\ 0 & 1 & \vdots & \\ \hdashline \underset{\sim}{\rho}_{(1)} & \underset{\sim}{\rho}_{(2)} & \vdots & 0 \end{pmatrix}-\sum_{i=1}^{5}\lambda_i^n\underset{\sim}{a}'_0\begin{pmatrix} 0 & 0 & \vdots & \underset{\sim}{0} \\ 0 & 0 & \vdots & \\ \hdashline \frac{1}{1-\lambda_i}A_iR_1 & \frac{1}{1-\lambda_i}A_iR_2 & \vdots & -A_i \end{pmatrix}$$

Let $f_{ij}(n)=P_r[X(n)=j \mid X(0)=i]$, $i$ transient state. Then, given $\underset{\sim}{a}'_0=(a_{01}, a_{02}, a_{03}, a_{04}, \ldots, a_{09})$, the absolute probability distribution at time $n$ is

$$\underset{\sim}{a}'_n=(a_{01}+b_1,\ a_{02}+b_2,\ b_3,\ b_4,\ b_5,\ b_6,\ b_7,\ b_8,\ b_9),$$

where

$$b_j=\sum_{i=3}^{9}b_{0i}f_{ij}(n).$$

Given in Table 5 are the absolute probability distributions at time $n$ given various transient states ($f_{ij}(n)$'s). This may be used to derive the absolute probability distributions of $X(n)$. This may also be used to compute the correlations of relatives as has been done in Horner (1956). As the procedure is rather straightforward, we will not go into detail, however.

**Table 5.** *Absolute probability distribution at time n given the transient states*

| | AA × Aa | aa × Aa |
|---|---|---|
| AA × AA | $\frac{1}{2}\left[1+\frac{x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^n}\times\left[\frac{\Delta_{n+2}}{\Delta_1}+\frac{\gamma_n(\sqrt{x})^{n+3}}{1+x+x^2}\right]$ | $\frac{1}{2}\left[1-\frac{x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^n}\times\left[\frac{\Delta_{n+2}}{\Delta_1}-\frac{\gamma_n(\sqrt{x})^{n+3}}{1+x+x^2}\right]$ |
| aa × aa | $\frac{1}{2}\left[1-\frac{x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^n}\times\left[\frac{\Delta_{n+2}}{\Delta_1}-\frac{\gamma_n(\sqrt{x})^{n+3}}{1+x+x^2}\right]$ | $\frac{1}{2}\left[1+\frac{x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^n}\times\left[\frac{\Delta_{n+2}}{\Delta_1}+\frac{\gamma_n(\sqrt{x})^{n+3}}{1+x+x^2}\right]$ |
| AA × Aa | $\frac{1}{4(1+x)^n}\left[(\sqrt{x})^n\varepsilon_n+\frac{2x}{\Delta_1}\Delta_{n-1}\right]$ | $\frac{1}{4(1+x)^n}\left[\frac{2x}{\Delta_1}\Delta_{n-1}-(\sqrt{x})^n\varepsilon_n\right]$ |
| aa × Aa | $\frac{1}{4(1+x)^n}\left[-(\sqrt{x})^n\varepsilon_n+\frac{2x}{\Delta_1}\Delta_{n-1}\right]$ | $\frac{1}{4(1+x)^n}\left[\frac{2x}{\Delta_1}\Delta_{n-1}+(\sqrt{x})^n\varepsilon_n\right]$ |
| Aa × AA | $\frac{1}{4(1+x)^n}\left[(\sqrt{x})^{n+1}\varepsilon_{n-1}+\frac{2x}{\Delta_1}\Delta_n\right]$ | $\frac{1}{4(1+x)^n}\left[\frac{2x}{\Delta_1}\Delta_n-(\sqrt{x})^{n+1}\varepsilon_{n-1}\right]$ |
| Aa × aa | $\frac{1}{4(1+x)^n}\left[-(\sqrt{x})^{n+1}\varepsilon_{n-1}+\frac{2x}{\Delta_1}\Delta_n\right]$ | $\frac{1}{4(1+x)^n}\left[\frac{2x}{\Delta_1}\Delta_n+(\sqrt{x})^{n+1}\varepsilon_{n-1}\right]$ |
| Aa × Aa | $\frac{1}{4(1+x)^n}\times\frac{4\Delta_n}{\Delta_1}$ | $\frac{1}{4(1+x)^n}\times\frac{4\Delta_n}{\Delta_1}$ |
| aa × AA | 0 | 0 |
| AA × aa | 0 | 0 |

| | Aa×AA | Aa×aa |
|---|---|---|
| AA×AA | $\frac{1}{2}\left[1+\frac{x+x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^n}\times\left[\frac{\Delta_{n+1}}{\Delta_1}+\frac{\gamma_{n-1}(\sqrt{x})^{n+2}}{1+x+x^2}\right]$ | $\frac{1}{2}\left[1-\frac{x+x^2}{1+x+x^3}\right]-\frac{1}{2(1+x)^n}\left[\frac{\Delta_{n+1}}{\Delta_1}-\frac{\gamma_{n-1}(\sqrt{x})^{n+3}}{1+x+x^2}\right]$ |
| aa×aa | $\frac{1}{2}\left[1-\frac{x+x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^n}\times\left[\frac{\Delta_{n+1}}{\Delta_1}-\frac{\gamma_{n-1}(\sqrt{x})^{n+2}}{1+x+x^2}\right]$ | $\frac{1}{2}\left[1+\frac{x+x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^n}\times\left[\frac{\Delta_{n+1}}{\Delta_1}+\frac{\gamma_{n-1}(\sqrt{x})^{n+3}}{1+x+x^2}\right]$ |
| AA×Aa | $\frac{1}{4(1+x)^n}\left[\frac{2\Delta_{n-2}}{\Delta_1}x+\varepsilon_{n-1}(\sqrt{x})^{n-1}\right]$ | $\frac{1}{4(1+x)^n}\left[\frac{2x}{\Delta_1}\Delta_{n-2}-\varepsilon_{n-1}(\sqrt{x})^{n-1}\right]$ |
| aa×Aa | $\frac{1}{4(1+x)^n}\left[\frac{2\Delta_{n-2}}{\Delta_1}x-\varepsilon_{n-1}(\sqrt{x})^{n-1}\right]$ | $\frac{1}{4(1+x)^n}\left[\frac{2x}{\Delta_1}\Delta_{n-2}+\varepsilon_{n-1}(\sqrt{x})^{n-1}\right]$ |
| Aa×AA | $\frac{1}{4(1+x)^n}\left[\frac{2\Delta_{n-1}}{\Delta_1}x+\varepsilon_n(\sqrt{x})^n\right]$ | $\frac{1}{4(1+x)^n}\left[\frac{2x}{\Delta_1}\Delta_{n-1}-\varepsilon_n(\sqrt{x})^n\right]$ |
| Aa×aa | $\frac{1}{4(1+x)^n}\left[\frac{2\Delta_{n-1}}{\Delta_1}x-\varepsilon_n(\sqrt{x})^n\right]$ | $\frac{1}{4(1+x)^n}\left[\frac{2x}{\Delta_1}\Delta_{n-1}+\varepsilon_n(\sqrt{x})^n\right]$ |
| Aa×Aa | $\frac{1}{4(1+x)^n}\times\frac{4\Delta_{n-1}}{\Delta_1}$ | $\frac{1}{4(1+x)^n}\times\frac{4\Delta_{n-1}}{\Delta_1}$ |
| aa×AA | 0 | 0 |
| AA×aa | 0 | 0 |

| | Aa×Aa | aa×AA |
|---|---|---|
| AA×AA | $\frac{1}{2}\left[1-\frac{1}{(1+x)^n}\cdot\frac{\Delta_{n+2}}{\Delta_1}\right]$ | $\frac{1}{2}\left[1+\frac{x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^{n-1}}\times\left[\frac{\Delta_{n+1}}{\Delta_1}+\frac{\gamma_{n-1}(\sqrt{x})^{n+2}}{1+x+x^2}\right]$ |
| aa×aa | $\frac{1}{2}\left[1-\frac{1}{(1+x)^n}\cdot\frac{\Delta_{n+2}}{\Delta_1}\right]$ | $\frac{1}{2}\left[1-\frac{x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^{n-1}}\times\left[\frac{\Delta_{n+1}}{\Delta_1}-\frac{\gamma_{n-1}(\sqrt{x})^{n+2}}{1+x+x^2}\right]$ |
| AA×Aa | $\frac{2x}{\Delta_1}\Delta_{n-1}$ | $\frac{1}{4(1+x)^{n-1}}\left[\frac{2x}{\Delta_1}\Delta_{n-2}+\varepsilon_{n-1}(\sqrt{x})^{n-1}\right]$ |
| aa×Aa | $\frac{2x}{\Delta_1}\Delta_{n-1}$ | $\frac{1}{4(1+x)^{n-1}}\left[\frac{2x}{\Delta_1}\Delta_{n-2}-\varepsilon_{n-1}(\sqrt{x})^{n-1}\right]$ |
| Aa×AA | $\frac{2x}{\Delta_1}\Delta_n$ | $\frac{1}{4(1+x)^{n-1}}\left[\frac{2x}{\Delta_1}\Delta_{n-1}+\varepsilon_n(\sqrt{x})^n\right]$ |
| Aa×aa | $\frac{2x}{\Delta_1}\Delta_n$ | $\frac{1}{4(1+x)^{n-1}}\left[\frac{2x}{\Delta_1}\Delta_{n-1}-\delta_n(\sqrt{x})^n\right]$ |
| Aa×Aa | $\frac{4\Delta_n}{\Delta_1}$ | $\frac{1}{4(1+x)^{n-1}}\times\frac{4\Delta_{n-1}}{\Delta_1}$ |
| aa×AA | 0 | 0 |
| AA×aa | 0 | 0 |

| | AA × aa |
|---|---|
| AA × AA | $\frac{1}{2}\left[1-\frac{x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^{n-1}}\left[\frac{\Delta_{n+1}}{\Delta_1}-\frac{\gamma_{n-1}(\sqrt{x})^{n+2}}{1+x+x^2}\right]$ |
| aa × aa | $\frac{1}{2}\left[1+\frac{x^2}{1+x+x^2}\right]-\frac{1}{2(1+x)^{n-1}}\left[\frac{\Delta_{n+1}}{\Delta_1}+\frac{\gamma_{n-1}(\sqrt{x})^{n+2}}{1+x+x^2}\right]$ |
| AA × Aa | $\frac{1}{4(1+x)^{n-1}}\left[\frac{2x}{\Delta_1}\Delta_{n-2}-\varepsilon_{n-1}(\sqrt{x})^{n-1}\right]$ |
| aa × Aa | $\frac{1}{4(1+x)^{n-1}}\left[\frac{2x}{\Delta_1}\Delta_{n-2}+\varepsilon_{n-1}(\sqrt{x})^{n-1}\right]$ |
| Aa × AA | $\frac{1}{4(1+x)^{n-1}}\left[\frac{2x}{\Delta_1}\Delta_{n-1}-\varepsilon_{n}(\sqrt{x})^{n}\right]$ |
| Aa × aa | $\frac{1}{4(1+x)^{n-1}}\left[\frac{2x}{\Delta_1}\Delta_{n-1}+\varepsilon_{n}(\sqrt{x})^{n}\right]$ |
| Aa × Aa | $\frac{1}{4(1+x)^{n-1}}\times\frac{4\Delta_{n-1}}{\Delta_1}$ |
| aa × AA | 0 |
| AA × aa | 0 |


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## Appendix 1

Lemma 1. Let $A$ be a $n\times n$ matrix. If $A$ is diagonable, then

$A=\sum_{j=1}^{l}\lambda_i A_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_l$ are the distinct eigenvalues of $A$; $\sum_{j=1}^{l}A_j=I_n$, $A_j^2=A_j$, and $A_iA_j=0$, $i\neq j$. Furthermore, $A_j=\prod_{\substack{i\\ i\neq j}}\frac{1}{(\lambda_j-\lambda_i)}(A-\lambda_i I_n)$, $j=1,2,\ldots,l$.

Proof. There exists a nonsingular matrix $P$ such that

$$P^{-1}AP=\text{Diag}\ (\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{r_1}, \underbrace{\lambda_2\, \lambda_2, \ldots, \lambda_2}_{r_2}, \ldots, \underbrace{\lambda_l, \ldots, \lambda_l}_{r_l}), =\lambda_1 E_1+\lambda_2 E_2+\cdots+\lambda_l E_l$$

where Diag $(x_1, x_2, \ldots, x_n)$ denotes a $n\times n$ diagonal matrix with diagonal elements $x_1, x_2, \ldots, x_n$, and

$$E_i=\text{Diag }(0,\ldots,0;\ \underbrace{1,1,\ldots,1}_{r_i};\ 0,\ldots,0),\ i=1,2,\ldots,l$$

Then,

$A=\sum_{j=1}^{l}\lambda_j PE_jP^{-1}=\sum_{j=1}^{l}\lambda_j A_j$, where $A_j=PE_jP^{-1}$. It is easy to check that $\sum_{j=1}^{l}A_j=I_n$, $A_j^2=A_j$ and $A_iA_j=0$ for $i\neq j$.

Furthermore, by pre- and post-multiplying $\prod_{i\neq j}\frac{1}{(\lambda_j-\lambda_i)}(A-\lambda_i I_n)$ by $P^{-1}$ and $P$ respectively, it is straightforward to show that

$P^{-1}\left\{\prod_{i\neq j}\frac{1}{(\lambda_j-\lambda_i)}(A-\lambda_i I_n)\right\}P=E_j$ so that $A_j=\prod_{i\neq j}\frac{1}{(\lambda_j-\lambda_i)}(A-\lambda_i I_n)$, $j=1,2,\ldots l$.

QED

Lemma 2. Let $Q=\begin{bmatrix}Q_{11} & 0\\ Q_{12} & 0\end{bmatrix}$ be $n\times n$ and $Q_{11}$ $r\times r$ diagonable. Then $Q$ is diagonable and $Q^n=\sum_{j=2}^{k}\mu_j\begin{bmatrix}B_j & 0\\ \frac{1}{\mu_j}Q_{12}B_j & 0\end{bmatrix}$, where $\mu_2, \mu_3, \ldots, \mu_k$ are the distinct non-zero eigenvalues of $Q_{11}$, and $B_j=\prod_{i\neq j}\frac{1}{(\mu_j-\mu_i)}(Q_{11}-\mu_i I_r)$.

Proof. The result follows immediately from $Q^n=\begin{bmatrix}Q_{11}^n & 0\\ Q_{12}Q_{11}^{n-1} & 0\end{bmatrix}$ and lemma 1.

QED

## Appendix 2

Section 2 is concerned only with finite Markov chain. It is, however, interesting to point out some connections between results of finite Markov chains with stationary transition probabilities and arbitrary Markov chains with stationary transition probabilities. For this purpose we put

$\rho_{ij}$=ultimate absorption probability into the $j$th closed set $C_j$ given initially $X(0)=i\varepsilon T$,

$\delta_{ij}(n)$=probability of first time absorption into $C_j$ at time $n$ given initially $X(0)=i\varepsilon T$.

Then it is easily seen that $\rho_{ij}$ and $\delta_{ij}(n)$ satisfy respectively,

$$\rho_{ij}=\sum_{v\varepsilon T}P_{iv}\rho_{vj}+\sum_{h\varepsilon C_j}P_{ih},\quad j=1,2,\ldots,k,\ i\varepsilon T,$$

$\delta_{ij}(n)=\sum_{v\varepsilon T}P_{iv}(n-1)\sum_{h\varepsilon C_j}P_{vh}$, $i\varepsilon T$, $j=1,2,\ldots,k$, where $P_{ij}$ and $P_{iv}(n-1)$ are the one-step and $(n-1)$ step transition probabilities respectively.

Thus, the probability of first time absorption into persistent states at time $n$ given initially $X(0)=i\varepsilon T$ is

$$\delta_i(n)=\sum_{j=1}^{k}\delta_{ij}(n)=\sum_{v\varepsilon T}P_{iv}(n-1)\sum_{j=1}^{k}\sum_{h\varepsilon C_j}P_{vh}$$
$$=\sum_{v\varepsilon T}P_{iv}(n-1)\{1-\sum_{h\varepsilon T}P_{vh}\}=\rho_i^{(n-1)}-\rho_i^{(n)}$$

where

$$\rho_i^{(n)}=P_r[X(n)\varepsilon T \mid X(0)=i],\quad i\varepsilon T.$$

If $\delta_i(n)$ is a p. d. f. over $n$, then $\delta_i(n)$ may be utilized to compute the mean absorption times $\mu_i$ and other moments of first absorption times of $i\varepsilon T$. From the above formular of $\delta_i(n)$, it is easy to see that $\delta_i(n)$ is a p. d. f. over $n$ if and only if $\lim_{n\to\infty}\rho_i^{(n)}=0$ (the limit exists as $\rho_i^{(n)}\leq\rho_i^{(n-1)}$), that is, the probability is zero that the chain will stay forever in $T$ given $X(0)=i\varepsilon T$. We now show in fact that the $\mu_i$ 's satisfy

$$\mu_i=1+\sum_{v\varepsilon T}P_{iv}\mu_v,\quad i\varepsilon T,$$

which is a well-known formular for mean absorption times.

We have, for every $i\varepsilon T$,

$$\mu_i=\sum_{n=1}^{\infty}n\delta_i(n)=\sum_{n=1}^{\infty}n[\rho_i^{(n-1)}-\rho_i^{(n)}]=\sum_{n=0}^{\infty}\rho_i^{(n)}=1+\sum_{n=1}^{\infty}\rho_i^{(n)},$$

and $$\rho_i^{(n)}=P_r[X(n)\varepsilon T \mid X_{(0)}=i]=\sum_{v\varepsilon T}P_{iv}P_r[X(n)\varepsilon T \mid X_{(1)}=v]=\sum_{v\varepsilon T}P_{iv}\rho_v^{(n-1)}.$$

Hence $\sum_{n=1}^{\infty}\rho_i^{(n)}=\sum_{v\varepsilon T}P_{iv}\sum_{n=0}^{\infty}\rho_v^{(n)}=\sum_{v\varepsilon T}P_{iv}\mu_v$, for every $i\varepsilon T$ so that, $\mu_i=1+\sum_{n=1}^{\infty}\rho_i^{(n)}$ $=1+\sum_{v\varepsilon T}P_{iv}\mu_v$, for all $i\varepsilon T$.

# 有限馬可夫鏈在遺傳學上的應用

## 論親子配偶系統中淘汰的影響力

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1. 本文首先對於時間穩定之有限馬可夫鏈 (Stationary finite Markov Chain) 提供一些數學公式藉以計算 (a) 過渡元素 (Transient states) 於時間 $n$ 及以前之被吸收入某一給定之閉合集 (Closed set) 之機率；(b) 過渡元素最終之被吸收入某一給定閉合集之機率；(c) 過渡元素初次於時間 $n$ 之被吸收入某一給定閉合集之機率；(d) 過渡元素之平均吸收入持久元素 (Persistent states) 之平均吸收時間及 (e) 過渡元素之被吸收入持久元素之變方 (Variance)。

2. 本文其次研究親子配偶系統中淘汰 (Selection) 之影響力；利用上述之公式以研究淘汰對於過渡型式 (Transient types) 之被吸收入二種配偶型 (Mating types) AA×AA 及 aa×aa 之各項機率，平均吸收時間及吸入時間之變方之效應。本文亦求出於時間 $n$ 時各配偶型之機率分布。本文所採用之方法基本爲行列之譜開展 (Spectral expansion)；故淘汰之效應基本上可由轉渡行列 (Transition matrix) 之固有根 (Latent roots 或 eigen-values) 表明之。