Bot. Bull. Academia Sinica **17**: 74-94 (1976)

DISTRIBUTION OF DEAD-HEARTS OF SUGAR CANE AND TRANSFORMATION OF DATA[1]

TSAN-LONG LIN

Department of Agonomy, National Taiwan University, Taipei, Taiwan, Republic of China

(Accepted for publication November 22, 1975)

Abstract

The negative binomial, Neyman type A and Poisson distributions were fitted to the observed distribution of the number per quadrat of dead-hearts of sugar cane. The negative binomial appeared to give a better fit.

Four types of data transformations were considered from the view of homogenizing within-treatment variances and reducing the correlation between means and variances. The family of transformation $(x+c)^b$ with x the number per quadrat of dead-hearts of sugar cane and c and b the parameters of the transformation estimated from samples under the constraints seemed better than the others. The logarithmic transformation with added one, $\ln(x+1)$, was not appriciative in some cases.

Introduction

The statistical analysis of "counts", such as the number of survivals at each plot before and after using the insectcide, or the number of dead-hearts of sugar cane in a plot observed at different occasions, is very complicated, since the normality and the variance-homogeneity conditions for experimental errors are not often fulfilled. For example, if the distribution of counts of some objects is Poisson, the variance equals to its mean, i.e., within-treatment variances change with treatment means. Therefore the more the treatment effects differ significantly, the more the within-treatment variances are heterogeneous. This difficulty may be overcome to some extent by using suitable transformed variables in the statistical analysis.

The objects of the present paper are (1) to study the distribution of the number per quadrat of dead-hearts of sugar cane, (2) to discuss if the commonly used transformations are applicable for the data in which the within-treatment variances are considerably heterogeneous.

(1) This work was supported by the National Science Council, Republic of China.

Data

An experiment with three treatments each with 36 quadrats was laid out at the farm of Taiwan Sugar Experiment Station in 1963. The area of each quadrat was 20×1.25 square metors. Field surveys began on the first of October 1963 and continued afterward for every 15 days interval until the end of January 1964. At each survey, the dead-heart in each quadrat was cut down and classified into three groups according to the causes of the dead-heart and was recorded separately. Group A conssisted of 24 sets of data with dead-hearts caused by cane borers regardless of the kind. Group B consisted of 9 sets of data with those caused by Chilo infuscatellus Snellen. Group C consisted of 27 sets of data with dead-hearts of all causes. Each set of data consisted of 36 observations with the same treatment at the same time of observation.

Distribution of Dead-Hearts

Sample means and mean squares are given in Table 1. In most of sets the variance seemed significantly larger the mean, which meant that the distribution of dead-hearts is clumped. In this study, two supper-dispersed distributions, negative binomial and Neyman type A, and Poisson distribution were fitted to to the data. The probability functions for these distributions are

$$\frac{\Gamma(x+k)}{\Gamma(k)x!}p^k q^x; \; q=1-p, \; x=0,1,\ldots. \qquad (1)$$

$$\frac{e^{-\lambda_1}\lambda_{2i}^x}{x!}\sum_{j=0}^{\infty}\frac{(\lambda_1 e^{-\lambda_2})^j j^x}{j!}; \; x=0,1,\ldots. \qquad (2)$$

and

$$\frac{e^{-\mu}\mu^x}{x!}; \; x=0,1,\ldots. \qquad (3)$$

respectively.

The maximum likelihood method was used to estimate unknown parameters. The parameter to be estimated in Poisson distribution is μ and its maximum likelihood estimate is the sample mean given by

$$\hat{\mu}=\frac{\sum_x f_x x}{\sum f_x}$$

where f_x is the observed frequency of the class x. The parameters to be estimated in the negative binomial distribution are k and p in the expression (1). Their maximum likelihood estimates can be obtained by solving following two equations with respect to k and p as has been suggested by Bliss and Fisher (1953):

Table 1. *Sample mean and sample variance*

Group	Time of obser-vtion	Treatment 1		Treatment 2		Treatment 3	
		Mean	Mean square	Mean	Mean square	Mean	Mean square
A	1	1.306	3.875	1.500	2.486	1.417	1.850
	2	4.667	22.343	4.250	16.536	6.139	21.152
	3	5.000	28.229	4.639	21.094	6.556	27.854
	4	2.389	11.444	3.861	15.494	3.083	6.364
	5	1.417	2.021	1.500	3.914	1.389	1.616
	6	1.028	1.456	1.611	7.959	2.250	10.364
	7	0.722	1.692	1.333	6.857	1.111	3.073
	8	0.611	1.387	0.389	0.530	0.389	0.530
B	1	0.583	0.764	0.806	1.018	0.722	0.778
	2	1.306	2.961	1.361	2.409	2.833	3.686
	3	0.361	0.809	0.306	0.333	*	*
	4	*	*	0.361	0.809	*	*
C	1	1.972	5.571	1.806	2.847	1.889	2.159
	2	10.389	41.559	10.444	33.740	13.722	46.035
	3	9.806	32.275	11.139	20.694	14.583	48.479
	4	6.583	12.650	9.222	27.206	10.083	29.336
	5	11.639	23.323	11.611	33.673	17.167	25.800
	6	17.389	52.416	21.750	67.830	23.056	55.368
	7	13.944	24.054	15.556	45.225	21.111	64.673
	8	7.250	9.507	7.306	17.304	7.250	14.879
	9	11.722	48.663	11.139	24.523	9.972	25.742

* $f_x=0$ for $x>1$.

$$\sum_x \frac{A_x}{k+x} + N \ln p = 0$$

$$\frac{Nk}{p} - \frac{T}{1-p} = 0$$

where $N=\Sigma f_x$, $T=\Sigma x f_x$ and $A_x = f_{x+1}+f_{x+2}+\ldots$. These equations are obtainable by differentiating the log-likelihood function with respect to k and p. Newton's method of solving equations was used after eliminating the second equation. The resultant equation was

$$S_c(k) = \sum_{x=0} \frac{A_x}{k+x} - N \ln \left(1+\frac{\bar{x}}{k}\right)$$

with derivative

$$DS_c(k) = -\Sigma \frac{A_x}{(k+x)^2} + \ln \frac{N\bar{x}}{k(k+\bar{x})}$$

To start the iteration, the moment estimate of k, as given by

$$k^{(0)}=\frac{\bar{x}^2}{s^2-\bar{x}}$$

was taken to be the initial value. In above, $\bar{x}$ and s^2 were the sample mean and the mean square, respectively. The iteration stopped as the abusolute value of $S_c(k)/DS_c(k)$ was less than 0.001.

For Neyman type A distribution, methods of obtaining maximum likelihood estimates have been suggested by Shenton (1949) and Douglas (1955). In this study, a computer program was set up as suggested below.

By differentiating log-likelihood function with respect to λ_1 and λ_2, the following two equations were obtained:

$$-N+\sum_{x=0} f_x e^{-\lambda_2}\sum_{j=0}\frac{(\lambda_1 e^{-\lambda_2})^j (j+1)^x}{j!}\Big/\sum_{j=0}\frac{(\lambda_1 e^{-\lambda_2})^j j^x}{j!}$$

$$\frac{T}{\lambda_2}-\sum_{x=0} f_x \lambda_1 e^{-\lambda_2}\sum_{j=0}\frac{(\lambda_1 e^{-\lambda_2})^j (j+1)^x}{j!}\Big/\sum_{j=0}\frac{(\lambda_1 e^{-\lambda_2})^j j^x}{j!}$$

Eliminating λ_1, the equation involving only λ_2 was obtained,

$$Al(\lambda_2)=-N+\sum_{x=0} f_x e^{-\lambda_2} V_x(\lambda_2)/U_x(\lambda_2)$$

where

$$\begin{aligned} U_x(\lambda_2) &= \sum_{j=0}(\bar{x}e^{-\lambda_2}/\lambda_2)^j j^x/j! \\ &= \sum_{j=1} a^j j^x/j! && \text{for } x>0 \\ &= e^a && \text{for } x=0 \\ V_x(\lambda_2) &= \sum_{j=0} a^j (j+x)^x/j && \text{for } x>0 \\ &= e^a && \text{for } x=0 \end{aligned}$$

and $a=\bar{x}e^{-\lambda_2}/\lambda_2$.

By the relation

$$\begin{aligned} U_{x+1}(\lambda_2) &= a\sum_{j=0} a^j (j+1)^j/j! \\ &= aV_x(\lambda_2) \end{aligned}$$

so

$$\begin{aligned} U'_x(\lambda_2) &= dU_x(\lambda_2)/d\lambda_2 \\ &= -(1+1/\lambda_2)U_{x+1}(\lambda_2) \end{aligned}$$

and

$$\begin{aligned} V'_x(\lambda_2) &= dV_x(\lambda_2)/d\lambda_2 \\ &= -\frac{1}{a}\left(1+\frac{1}{\lambda_2}\right)[U_{x+2}(\lambda_2)-U_{x+1}(\lambda_2)], \end{aligned}$$

the derivative of $Al(\lambda_2)$, denoted by $Ald(\lambda_2)$, was given by

$$Ald(\lambda_2)=-Al(\lambda_2)-N-e^{-\lambda_2}\left(1+\frac{1}{\lambda_2}\right)\sum_{x=0} f_x$$

$$\cdot\left\{\frac{U_{x+2}(\lambda_2)-U_{x+1}(\lambda_2)}{aU_x(\lambda_2)}-\frac{[U_{x+1}(\lambda_2)]^2}{a[U_x(\lambda_2)]^2}\right\}.$$

Newton's method of solving equations was used in obtaing the maximum likelihood estimate of λ_2 with successive application of

$$\lambda_2^{(n)}=\lambda_2^{(n-1)}-Al(\lambda_2^{(n-1)})/Ald(\lambda_2^{(n-1)}) \qquad (4)$$
$$\lambda_1^{(n)}=\bar{x}/\lambda_2^{(n)}.$$

The iteration stopped as the second term of the right side in equation (4) was less than 0.001. The staring values were obtained by solving

$$\lambda_1^{(0)}\lambda_2^{(0)}=\bar{x}$$

and

$$\lambda_1^{(0)}\lambda_2^{(0)}(1+\lambda_2^{(0)})=s^2$$

The recurrent formula used for calculating frequency at $x+1$ was

$$\hat{f}_{x+1}=\frac{\hat{\lambda}_1\hat{\lambda}_2e^{-\hat{\lambda}_2}}{x+1}\sum_{g=0}^{x}\frac{\hat{\lambda}_2^{x-g}}{(x-g)!}\hat{f}_g.$$

Results were listed in Appendix I.

On testing for goodness-of-fit of the theoretical distribution to the observed, the X^2 statistic as defined below was used:

$$X^2=(f_x-\hat{f}_x)^2/\hat{f}_x$$

where f_x = observed frequency in the xth class,

$\hat{f}_x$ = expected frequency in the xth class.

The X^2 statistic was evaluated by keeping $\hat{f}_x \geqq 1$ as suggested by Cochran (1954). Though the method provided by Nass, as suggested by Pahl (1969) was used, the results showed little difference with the above mentioned, only the latter was shown in Appendix II. Since the fit to Poisson distribution was, in general, very poor as could be seen from the Table 1, X^2-values corresponding to this distribution were not given.

In most cases discrepancies between two supperdispersed theoretical distributions and the observed were negligible. By judging from X^2 values, though in some sets Neyman A series showed a better fit than the negative binomial, fit in group A treatment 1, the 3rd set, treatment 3, the 7th set and group B treatment 2, the 4th set seemed to be extremely poor. Such an exception was not observed in the fit of the negative binomial distribution.

Transformation of Data

Often the treatment applied in the field experiment of the kind been described, the persistence of the treatment effect is of major interest. The model in the analysis of variance is

$$y_{ijk}=u+a_i+t_j+(at)_{ij}+e_{ijk}$$

where a_i, t_j and $(at)_{ij}$ designate the ith treatment effect, the jth time effect, and the effectiveness of the ith treatment at the jth time period, respectively. For the analysis of variance to be applicable, at least conditions of variances homogeniety and homoscedasticity must be nearly fulfilled.

Since as been shown, the observed distribution was super-dispersed, the mean and the variance might be highly correlated and the heterogeniety of within treatment variances was expected. The correlation coefficient between sample means and sample variances, designated by r, and the M statistic for testing homogeniety of variances are given in Table 2. The M statistic is defined by

$$M=[n\ln(\sum_i\sum_j S^2_{ij}n_{ij}/n)-\sum_i\sum_j n_{ij}\ln(S^2_{ij})]/[1+(\sum_i\sum_j n_{ij}^{-1}-n^{-1})/3(\sum_i\sum_j(1)-1)]$$

$$S^2_{ij}=\sum_{k=1}(y_{ijk}-\bar{y}_{ij})^2/n_{ij}$$

and $n=\sum_i\sum_j n_{ij}$ and n_{ij} is the degrees of freedom of the (ij)th mean square S^2_{ij}.

Table 2. *Correlation coefficient between means and variances, and the M statistic for testing homogeniety of variances*

Group	A	B	C
Correlation (r)	0.952	0.925	0.885
M statistic	507.03	84.91	244.50
DF of M	23	8	26

The sample mean and the sample variance are seen to be highly correlated and the heterogeneity of variances is evident.

Many types of transformations have been proposed for stabilizing the variance of negative binomial variables, of which following three are commonly used:

$y=\ln(x+c)$ $c=k/2$ by Anscombe (1948)

$y=\text{arcsinh}((x+c)/b)^{1/2}$

$c=3/8$ for large mean and large (number of observation — mean) and $b=k-2c$, by Anscombe (1948)

$y=\text{arcsinh}(cx)^{1/2}$ $c=k$ by Beall (1942).

In each transformation, x designates the count per quadrat and y the transformed variable.

To the three types of transformations, the following type of the transformation first proposed by Moore and Tukey (1954) and extensively studied by Box and Cox (1964) is added:

$$(x+c)^b.$$

In this study c or c and b in the transformation were regarded as parameters and were estimated from samples under the constraints: (1) The abusolute value of correlation coefficient between means and variances of transformed variables was less than 0.1 and (2) The minimum of M statistic satisfying (1) was attained.

Transformations were evaluated for $c>0.0$ and $b>0.0$ for $\text{arcsinh}\,((x+c)/b)^{1/2}$ transformation and $-4\leqslant b\leqslant 4$ for $(x+c)^b$ transformation at intervals of 0.1 in order to obtain initial estimates. The simplex procedure (O'Neill, 1971) was then used to allocate the point with minimum of M statistic. Since the transformation so obtained gave r greater than 0.1 in absolute value in several cases, transformations at the neighbourhood of the minimum M estimates were evaluated at intervals of 0.01. Results are shown in the Table 3. Values of M and r at the neighbourhood of the optimum points were evaluated at intervals of 0.01 and 0.05 and are given in Appendix III and IV.

It is seen from the Appendix IV that the values of M statistic change slowly at the neighbourhood of the minimum of M. At the neighbourhood, change of the values of r is critical. It is also seen from Table 3 that in $(x+c)^b$ transformation the optimum point appears in a rather narrow range, i.e., in $0.25\leq b\leq 0.4$ and $c\leq 0.1$. In $\text{arcsinh}\,((x+c)/b)^{1/2}$ transformation, the change of M around the minimum M is almost inert to the change of b-values.

Discussion

The observed distribution of dead-hearts of sugar cane is seen to be clumped and shows a good fit to the negative binomial distribution.

To compare results in Table 3 with those of Table 2, it is seen that the heterogeniety of error variances is greatly reduced by the data transformation with suitablly chosen parameters. If the distribution of the M statistic can be approximated by chi-square distribution, the probability of obtaining larger M values than those listed in Table 3 is about 0.25 to 0.10 for group B and group C, and about 0.025 to 0.01 for group A.

Many types of transformations have been proposed for stabilizing the variance of negative binomial distributions. Most widely applicable one is the logarithm of $(x+1)$. This type of transformation is not appreciative in the cases studied, since, as seen in Table 4, it gave both large values of M statistic and r.

When several sets of negative binomial data are available, it has also been

Table 3. *Estimate of c or (c, b) at which M is minimum or minimum among the transformations with $|r|<0.1$*

Group	Transformation	c	b	M	r
A	$(x+c)^b$	0.000*	0.294	39.151	0.096
	arcsinh $(cx)^{1/2}$	1.077		41.054	0.101
		1.088		41.055	0.098
	ln $(x+c)$	0.167		41.513	0.090
	arcsinh $((x+c)/b)^{1/2}$	0.000*	0.907	41.062	0.097
B	$(x+c)^b$	0.000*	0.808	11.735	0.184
		0.000*	0.258	11.973	0.093
	arcsinh $(cx)^{1/2}$	1.672		12.095	0.171
		3.049		12.237	0.099
	ln $(x+c)$	0.104		12.188	0.155
		0.064		12.314	0.091
	arcsinh $((x+c)/b)^{1/2}$	0.000*	0.606	12 067	0.170
		0.010	0.180	12.275	0.097
C	$(x+c)^b$	0.116	0.397	29.348	−0.023
	arcsinh $(cx)^{1/2}$	0.020		29.755	−0.056
	ln $(x+c)$	2.740		38.410	−0.107
		2.770		38.413	−0.098
	arcsinh $((x+c)/b)^{1/2}$	0.002	50.700	29.686	−0.015

* To avoid computational difficulty, 0.000001 is substituted instead of zero.

suggested to use common k as a estimate in transformations of the kind given in the Table 4. These transformations with common k, designated by k_c in the table, sometimes give results very near to the optimum, but may provide poor results.

The parameter(s) in the transformation must be estimated from the sample directly under some well defined constraints. Also, the transformation of the

Table 4. *Results of transformations of ln $(x+1)$, ln $(x+0.5k_c)$, $\sinh^{-2}(k_c x)^{1/2}$ and $\sinh^{-1}((x+0.375)/(k_c-0.750))^{1/2}$*

Group	Common $k(k_c)$		ln $(x+1)$	ln $(x+0.5k_c)$	$\sinh^{-1}(k_c x)^{1/2}$	$\sinh^{-1}((x+0.375)/(k_c-0.750))^{1/2}$
A	1.548	M	84.20	71.06	41.70	65.76
		r	0.661	0.601	−0.002	0.569
B	4.319	M	21.47	32.60	12.41	23.25
		r	0.628	0.761	0.065	0.664
C	8.182	M	66.91	42.47	580.43	35.82
		r	−0.766	0.167	−0.930	−0.276

type $(x+c)^b$ always give better results than those of the others, the merit of the transformation can only be appreciative for the properly chosen values of parameters. If the parameters chosen are improper results may be worse than those of untransformed. This can be seen from Appendix IV. This is true for any other types of transformations.

Acknowledgment

The auther is greatful to Mr. Wen-yi Cheng of Taiwan Sugar Research Institute for his kind allowance to use the data described in this paper.

Literature Cited

ANSCOMBE, F. J. 1948. The transformation of Poisson, binomial and negative binomial data. Biometrika **35**: 246-254.

BEALL, G. 1942. The transformation of data from entomological field experiments so that the analysis of variance becemes applicable. Biometrika, **32**: 243-262.

BLISS, C. I. and R. A. FISHER. 1953. Fitting the negative binomial distribution to biological data. Biometrics **9**: 176-199.

BLISS, C. I. and A. R. G. OWEN. 1958. Negative binomial distributions with a commom k. Biometrika **45**: 37-58.

BOX, G. E. P., and COX, D. R. 1964. An analysis of transfprmations. Jour. Royal Statist. B **26**: 211-343. Discussion 244-252.

COCHRAN, W. G. 1954. Some mothods of strengthening the common x^2-tests. Biometrics **10**: 417-451.

DOUGLAS, J. B. 1955. Fitting the Neyman type A (two parameters) contagious distribution. Biometrics. **11**: 149-173.

MOORE, G. and TUKEY, J. W. 1954. Answer to Queries. Biometric **10**: 562-568.

O'NEILL, R. (1971). Function minimization using a simplex procedure. Applied Statistics **20**: 338-345.

PAHL, P. J. 1969. On testing for goodness-of-fit of the negative binomial distribution when expectations are small. Biometrics **25**: 143-151.

SHENTON, L. R. 1949. On the efficiency of the method of moments and Neyman's type A distribution. Biometrika **36**: 450-454.

甘蔗枯心之分佈及變值轉換

林　燦　隆

國立臺灣大學農藝學系

由六十組枯心調查資料，知枯心分佈適合負二項分佈。雖在某些組中，Neyman A 型分佈的適合性較負二項分佈為優，但在某些情形下適合性甚爲不良。

爲穩定變方，常用之負二項變值轉換方法，有 log $(x+c)$, $\sinh^{-1}\sqrt{\frac{x+c}{b-2c}}$, $\sinh^{-1}\sqrt{cx}$。本研究除此之外，另加 $(x+c)^b$ 的轉換。此時，x 爲每小區的枯心數。

在變方均質而平均與變方之相關爲最小之條件下，求 c 或 c 與 b，所得結果如表(三)。用 $\ln(x+1)$, $\ln(x+k_c/2)$, $\sinh^{-1}\sqrt{k_c x}$, k_c 爲共同 k (Common k) 時之結果如表(四)，其結果不儘理想。

Appendix I. *Maximum likelihood estimates of parametrs*

Group	Time of Obser-vation	Treatment 1				Treatment 2				Treatment 3			
		NB**		NA**		NB		NA		NB		NA	
		k	p	λ_1	λ_2	k	p	λ_1	λ_2	k	p	λ_1	λ_2
A	1	1.173	0.473	1.609	0.812	2.079	0.581	2.312	0.640	5.151	0.784	4.853	0.292
	2	1.112	0.192	1.693	2.748	1.506	0.262	1.820	2.336	3.091	0.335	4.033	1.522
	3	0.980	0.164	2.186	2.287	1.019	0.180	1.826	2.540	1.814	0.217	2.489	2.634
	4	0.716	0.231	1.369	1.745	0.968	0.200	1.769	2.182	3.096	0.501	3.863	0.778
	5	3.900	0.733	4.590	0.311	0.866	0.366	1.289	1.164	10.890	0.887	11.753	0.118
	6	2.745	0.728	2.964	0.367	0.431	0.211	0.392	1.806	0.686	0.234	1.393	1.615
	7	0.427	0.372	0.552	1.307	0.247	0.156	0.528	2.523	0.518	0.313	0.828	1.342
	8	0.611	1.387	0.748	0.818	1.070*	0.734*	1.071*	0.364*	1.009*	0.722*	1.066*	0.365*
B	1	1.729	0.748	1.730	0.337	2.344	0.744	3.051	0.264	13.419	0.940	12.021	0.060
	2	0.885	0.390	1.071	1.219	1.544	0.522	1.764	0.772	10.171	0.732	10.463	0.271
	3	0.233	0.392	0.324	1.113	3.993*	0.929*	3.087*	0.099*	—	—	—	—
	4					0.233	0.392	0.325	1.113	—	—	—	—
C	1	1.662	0.457	2.206	0.894	3.001	0.624	3.234	0.558	17.319	0.902	18.254	0.103
	1	4.048	0.280	4.609	2.254	5.865	0.360	6.580	1.587	5.349	0.281	5.960	2.302
	3	4.387	0.309	5.024	1.952	13.287	0.544	13.540	0.823	8.315	0.363	8.946	1.630
	4	8.325	0.558	9.099	0.724	5.123	0.357	5.786	1.594	5.614	0.358	6.277	1.606
	5	11.882	0.505	12.294	0.947	5.209	0.310	5.831	1.991	38.219	0.690	39.155	0.438
	6	9.693	0.358	10.270	1.693	9.071	0.294	9.662	2.251	16.773	0.421	17.212	1.340
	7	21.451	0.606	22.315	0.625	7.737	0.332	8.264	1.882	11.487*	0.342*	12.487*	1.271*
	8	24.778	0.774	24.223	0.259	5.084	0.410	5.830	1.253	7.567	0.511	8.376	0.866
	9	5.417	0.316	5.890	1.990	10.863	0.494	11.676	0.954	7.820	0.440	8.583	1.169

* Number of classes $\leqslant 3$ or $\geqslant 45$. Only correct to two decimal places.
** NB: Negatine Binomial Distribution NA: Neyman A Dislribution.

Appendix II. *Chi-squares for testing goodness-of-fit*

Group	Time of Observation	Treatment 1				Treatment 2				Treatment 3			
		NB		NA		NB		NA		NB		NA	
		X^2	DF	X^2	DF	X^2	DF	X^2	DF	X^2	DF	X^2	DF
A	1	2.49	3	2.16	3	0.81	3	0.89	3	1.66	3	1.44	2
	2	8.79	11	7.67	10	11.48	9	10.46	9	16.10	11	15.63	11
	3	16.67	11	27.53	10	8.78	10	12.29	10	9.05	13	9.62	12
	4	3.65	6	6.68	6	9.63	9	13.25	8	4.05	6	4.90	6
	5	2.15	4	1.94	2	2.27	5	9.07	4	0.98	2	1.21	2
	6	2.02	2	2.93	2	6.59	5	9.20	4	7.10	6	14.49	5
	7	3.08	2	0.93	2	2.96	4	3.56	4	6.36	4	31.73	3
	8	3.20	2	2.39	1	*	*	*	*	*	*	*	*
B	1	0.26	1	0.85	1	1.27	1	2.12	1	0.24	1	0.61	1
	2	1.12	3	0.49	3	1.18	3	0.60	2	11.27	5	9.34	4
	3	0.44	2	5.30	1	*	*	*	*				
	4					3.19	1	9.59	1				
C	1	3.44	4	3.49	5	1.29	4	1.18	3	1.47	3	1.54	3
	2	14.46	15	20.73	15	25.17	13	28.84	13	16.91	18	15.94	17
	3	14.04	15	13.41	14	7.98	13	11.65	13	11.81	16	12.48	15
	4	4.83	10	4.86	10	11.37	14	7.32	13	15.60	14	15.72	14
	5	20.75	11	20.61	14	22.15	15	19.80	15	15.16	15	15.41	15
	6	20.39	18	20.45	17	26.38	21	26.77	21	12.35	20	13.48	19
	7	12.09	14	12.25	14	21.77	18	24.33	18	30.96	20	29.82	20
	8	18.93	10	24.51	9	10.04	13	9.66	11	6.17	12	5.62	10
	9	20.53	17	22.07	16	10.20	13	10.62	13	5.83	13	6.59	14

* Zero degrees of freedom

Appendix III. *The change of M and r statistics at the neighbourhood of the optimum point*

Group A

$(x+c)^b$ transformation

b \ c		0.00*	0.01	0.02	0.03	0.04	0.05
0.274	M	36.69	41.80	44.18	46.36	48.40	50.34
	r	0.002	0.273	0.342	0.387	0.422	0.449
0.284	M	39.29	42.69	45.37	47.74	49.93	51.99
	r	0.049	0.305	0.370	0.413	0.445	0.471
0.294	M	39.15	43.76	46.72	49.28	51.61	53.78
	r	0.096	0.337	0.398	0.438	0.468	0.493
0.304	M	39.281	45.00	48.23	50.97	53.43	55.72
	r	0.143	0.367	0.424	0.462	0.490	0.513
0.314	M	39.68	46.42	49.91	52.81	55.40	57.79
	r	0.189	0.396	0.449	0.485	0.511	0.533

arcsinh $(cx)^{1/2}$ transformation

c	1.05	1.06	1.07	1.08	1.09	1.10	1.11	1.12	1.13	1.14
M	41.06	41.06	41.06	41.05	41.06	41.06	41.06	41.06	41.07	41.07
r	0.108	0.106	0.103	0.100	0.098	0.095	0.092	0.090	0.087	7.085

In $(x+c)$ transformation

c	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20	0.21
M	42.42	42.07	41.81	41.64	41.54	41.51	41.54	41.62	41.75	41.91
r	−0.029	−0.002	0.023	0.047	0.070	0.092	0.112	0.132	0.151	0.151

arcsinh $((x+c)/b)^{1/2}$ transformation

c \ b		0.87	0.88	0.89	0.90	0.91	0.92	0.93	0.94	0.95	0.96
0.000*	M	41.08	41.07	41.07	41.06	41.06	41.06	41.06	41.06	41.07	41.07
	r	0.084	0.088	0.091	0.094	0.097	0.100	0.103	0.106	0.109	0.112
0.010	M	41.75	41.78	41.82	41.86	41.90	41.94	41.98	42.02	42.06	42.10
	r	0.184	0.187	0.190	0.193	0.196	0.199	0.201	0.204	0.207	0.210
0.020	M	42.44	42.49	42.54	42.60	42.65	42.71	42.77	42.82	42.88	42.94
	r	0.225	0.228	0.231	0.233	0.236	0.239	0.242	0.244	0.247	0.250

* 0.000001

Group B

$(x+c)^b$ transformation

b \ c		0.00*	0.01	0.02	0.03	0.04	0.05	0.06
0.248	M	12.07	11.92	12.12	12.34	12.57	12.80	13.04
	r	0.077	0.218	0.262	0.294	0.319	0.341	0.360
0.258	M	11.97	11.96	12.19	12.44	12.69	12.94	13.19
	r	0.093	0.233	0.276	0.308	0.333	0.354	0.373
0.268	M	11.89	12.00	12.28	12.55	12.83	13.09	14.36
	r	0.111	0.248	0.291	0.322	0.347	0.368	0.386
0.278	M	11.82	12.07	12.38	12.68	12.97	13.26	13.53
	r	0.128	0.264	0.306	0.336	0.360	0.381	0.399
0.288	M	11.78	12.15	12.50	12.82	13.13	13.44	13.73
	r	0.146	0.279	0.321	0.350	0.374	0.395	0.412

arcsinh $(cx)^{1/2}$ transformation

c	2.99	3.00	3.01	3.02	3.03	3.04	3.05	3.06	3.07	3.08
M	12.23	12.23	12.23	12.23	12.23	12.23	12.24	12.24	12.24	12.24
r	0.101	0.101	0.101	0.100	0.100	0.100	0.099	0.098	0.098	0.098

ln $(x+c)$ transformation

c	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13
M	12.52	12.40	12.31	12.25	12.22	12.20	12.20	12.20	12.21	12.24
r	0.052	0.073	0.091	0.109	0.125	0.141	0.155	0.169	0.183	0.195

arcsinh $((x+c)/b)^{1/2}$ transformation

c \ b		0.14	0.15	0.16	0.17	0.18	0.19	0.20	0.21	0.22	0.23
0.010	M	12.36	12.34	12.32	12.29	12.28	12.26	12.24	12.23	12.32	12.20
	r	0.077	0.083	0.088	0.093	0.097	0.102	0.106	0.111	0.115	0.119
0.020	M	12.26	12.24	12.23	12.22	12.21	12.20	12.19	12.18	12.18	12.17
	r	0.103	0.108	0.113	0.118	0.122	0.127	0.131	0.136	0.140	0.144
0.030	M	12.21	12.20	12.19	12.18	12.18	12.18	12.17	12.17	12.17	12.17
	r	0.124	0.129	0.134	0.138	0.143	0.147	0.152	0.156	0.160	0.164
0.040	M	12.19	12.18	12.18	12.18	12.18	12.18	12.28	12.18	12.19	12.19
	r	0.142	0.147	0.151	0.156	0.161	0.165	0.169	0.173	0.177	0.181

* 0.000001

Group C

$(x+c)^b$ transformation

b \ c		0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16
0.377	M	31.20	30.78	30.44	30.17	29.96	29.79	29.66	29.56	29.49	29.44
	r	−0.261	−0.235	−0.211	−0.188	−0.166	−0.146	−0.127	−0.109	−0.092	−0.076
0.387	M	30.35	30.06	29.83	29.66	29.54	29.45	29.40	29.37	29.36	29.37
	r	−0.195	−0.169	−0.145	−0.123	−0.102	−0.083	−0.065	−0.048	−0.032	−0.016
0.397	M	29.78	29.60	29.48	29.40	26.36	29.35	29.36	29.39	29.44	29.51
	r	−0.127	−0.102	−0.080	−0.059	−0.039	−0.021	−0.004	0.012	0.027	0.042
0.407	M	29.47	29.40	29.37	29.38	29.41	29.46	29.54	29.63	29.73	29.85
	r	−0.061	−0.037	−0.016	0.004	0.022	0.039	0.055	0.070	0.084	0.097
0.417	M	29.42	29.44	29.50	29.58	29.68	29.79	29.92	30.07	30.22	30.38
	r	0.005	0.027	0.046	0.065	0.081	0.097	0.111	0.125	0.138	0.150

arcsinh $(cx)^{1/2}$ transformation

c	0.00*	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
M	32.56	30.13	29.76	30.94	32.03	33.67	36.22	38.67	41.26	43.94
r	0.267	0.099	−0.056	−0.187	−0.295	−0.384	−0.458	−0.519	−0.569	−0.611

ln $(x+c)$ transformation

c	2.73	2.74	2.75	2.76	2.77	2.78	2.79	2.80	2.81
M	38.41	38.41	38.41	38.41	38.41	38.41	38.42	38.42	38.43
r	−0.109	−0.107	−0.104	−0.101	−0.098	−0.096	−0.093	−0.090	−0.088

arcsinh $((x+c)/b)^{1/2}$ transformation

c \ b		50.66	50.67	50.68	50.69	50.70	50.71	50.72	50.73	50.74	50.75
0.000*	M	29.74	29.74	29.74	29.74	29.74	29.74	29.74	29.74	29.74	29.74
	r	−0.049	−0.049	−0.049	−0.046	−0.049	−0.049	−0.049	−0.049	−0.049	−0.049
0.010	M	29.82	29.82	29.82	29.82	29.82	29.82	29.82	29.82	29.82	29.82
	r	0.039	0.039	0.039	0.039	0.040	0.040	0.040	0.040	0.040	0.040
0.020	M	30.06	30.06	30.06	30.06	30.06	30.06	30.06	30.06	30.06	30.06
	r	0.074	0.074	0.074	0.074	0.074	0.074	0.074	0.074	0.074	0.075
0.030	M	30.31	30.31	30.31	30.31	30.31	30.32	30.32	30.32	30.32	30.32
	r	0.099	0.099	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
0.040	M	30.57	30.57	30.57	30.58	30.58	30.58	30.58	30.58	30.57	30.58
	r	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120

* 0.000001

Appendix IV. *Values of M statistic at the neighbourhood of the minimum M evaluated at intervals of 0.05*

Group A $(x+c)^b$ transformation

b \ c	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
−0.006	133.8	50.5	43.8	41.7	41.7	42.8	44.5	46.6	49.0	51.6	54.3	57.1
0.004	108.7	45.2	41.5	41.5	42.9	45.1	47.7	50.7	53.8	56.9	60.2	63.4
0.094	86.1	41.7	41.1	43.0	45.8	49.1	52.7	56.3	60.0	63.8	67.5	71.2
0.144	66.8	40.5	42.8	56.5	50.7	55.0	59.4	63.7	68.0	72.2	76.3	80.4
0.194	52.0	41.9	46.9	52.3	57.6	62.9	68.0	72.9	77.7	82.3	86.8	91.1
0.224	42.5	46.2	53.6	60.5	66.9	72.9	78.7	84.1	89.3	94.2	99.0	103.6
0.294	39.2	53.8	63.3	71.4	78.6	85.3	91.5	97.3	102.8	108.0	113.0	117.7
0.344	42.5	64.9	76.0	85.0	92.9	100.0	106.6	112.7	118.4	123.7	128.8	133.6
0.394	52.5	79.7	91.8	401.5	109.8	117.2	124.0	130.2	136.0	141.4	146.5	151.4
0.444	69.0	98.2	111.0	121.0	129.0	136.9	143.7	149.9	155.7	161.1	166.1	171.0
0.494	91.3	120.4	133.3	143.3	151.7	159.1	165.8	171.9	177.6	182.8	187.7	192.4
0.544	118.8	146.3	158.8	168.5	176.7	183.8	190.2	196.0	201.5	206.5	211.2	215.7
0.594	150.8	175.6	187.4	196.5	204.2	210.9	216.9	222.4	227.5	232.2	236.6	240.8
0.644	186.6	208.1	218.8	227.1	534.1	240.3	245.8	250.6	255.5	259.9	263.6	267.7
0.664	225.6	243.5	252.9	260.2	266.5	272.0	276.9	281.4	285.5	289.4	293.0	296.4

Transformations with $|r|<0.1$ are surrounded with segments.

Appendix IV. (*Continued*)

Group B $(x+c)^b$ transformation

b \ c	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
−0.008	15.8	12.4	12.2	12.3	12.6	13.0	13.5	14.0	14.6	15.2	15.8	16.4
0.058	15.1	12.2	12.2	12.5	13.0	13.5	14.1	14.7	15.4	16.0	16.7	17.4
0.108	14.3	12.1	12.3	12.9	13.5	14.2	14.9	15.6	16.4	17.1	17.8	18.5
0.158	13.5	12.1	12.7	13.4	14.2	15.0	15.9	16.7	17.5	18.3	19.1	19.9
0.208	12.6	12.4	13.3	14.3	15.2	16.2	17.1	18.0	18.9	19.8	20.7	21.5
0.258	12.0	12.9	14.2	15.3	16.5	17.6	18.6	19.6	20.6	21.5	22.4	23.3
0.308	11.7	13.8	15.4	16.7	18.0	19.2	20.4	21.4	22.5	23.5	24.4	25.3
0.358	12.0	15.1	16.9	18.5	19.9	21.2	22.4	23.6	24.6	25.7	26.7	27.6
0.408	13.0	16.9	19.0	20.7	22.2	23.6	24.8	26.1	27.1	28.2	29.2	30.1
0.458	14.7	19.1	21.4	23.3	24.8	26.3	27.6	28.8	29.9	31.0	32.0	33.0
0.508	17.2	22.0	24.4	26.3	28.0	29.4	30.7	32.0	33.1	34.2	35.2	36.1
0.558	20.5	25.5	27.9	29.9	31.5	33.0	34.3	35.5	36.6	37.6	38.6	39.5
0.608	24.8	29.6	32.0	33.9	35.5	37.0	38.2	39.4	40.5	41.5	42.4	43.3
0.658	29.9	34.4	36.7	38.5	40.1	41.4	42.6	43.7	44.8	45.7	46.6	47.4
0.708	35.8	39.8	42.0	43.7	45.1	46.4	47.5	48.6	49.4	50.3	51.1	51.8

Appendix IV. (*Continued*)

Group C $(x+c)^b$ transformation

b \ c	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
−0.097	948.4	261.7	193.2	156.3	132.2	114.9	101.9	91.6	83.4	76.6	70.9	66.2
0.147	613.7	183.0	163.3	111.1	94.6	82.9	74.1	67.2	61.7	57.2	53.5	50.4
0.197	376.7	124.0	93.8	77.5	66.9	59.4	53.8	49.5	46.1	43.3	41.2	39.5
0.247	220.4	82.0	63.7	53.9	47.7	43.3	40.2	37.8	36.1	34.8	33.8	33.0
0.297	124.0	54.0	44.0	38.9	35.8	33.7	32.4	31.5	31.0	30.6	30.5	30.5
0.347	69.0	37.5	33.1	31.1	30.2	29.7	29.7	29.8	30.1	30.5	31.0	31.6
0.397	41.1	30.3	29.4	29.5	29.9	30.6	31.4	32.2	33.1	34.0	34.9	35.8
0.447	30.9	30.3	31.5	32.9	34.2	35.5	36.8	38.1	39.3	40.6	41.7	42.9
0.497	32.2	36.1	38.5	40.5	42.3	44.0	45.6	47.1	48.5	49.9	51.2	52.5
0.547	41.0	46.5	49.3	51.6	53.6	55.5	57.2	58.8	60.3	61.7	63.1	64.4
0.597	55.0	60.5	63.3	65.7	67.7	69.5	71.2	72.8	74.3	75.7	77.0	78.3
0.647	72.4	77.3	80.0	82.2	84.1	85.8	87.4	88.6	90.3	91.6	92.8	94.1
0.697	92.5	96.5	98.9	100.8	102.5	104.0	105.4	106.8	108.0	109.2	110.3	111.4
0.747	114.5	117.6	119.6	121.2	122.6	123.9	125.1	126.3	127.4	128.4	129.4	130.3
0.797	137.9	140.3	141.8	143.1	144.3	145.3	146.3	147.2	148.1	149.0	149.8	150.5

Appendix IV. (*Continued*)

Group A $\sinh^{-1}((x+c)/b)^{1/2}$ transformation

c \ c	0.67	0.72	0.77	0.82	0.87	0.92	0.97	1.02	1.07	1.12	1.17	1 22
0.000	41.6	41.4	41.2	44.1	41.0	41.0	41.0	41.1	41.2	41.3	41.4	41.5
0.050	42.9	43.3	43.7	44.1	44.5	45.0	45.4	45.8	46.3	46.7	47.2	47.6
0.100	45.7	46.3	46.9	47.5	48.0	48.7	49.2	49.8	50.4	51.0	51.6	52.1
0.150	48.8	49.5	50.2	50.9	51.6	52.3	53.0	53.7	54.3	55.0	55.6	56.3
0.200	52.0	52.8	53.6	54.4	55.1	55.9	56.6	57.4	58.1	58.8	59.5	60.2
0.250	55.2	56.0	56.9	57.8	58.6	59.4	60.2	61.0	61.8	62.5	63.3	64.2
0.300	58.4	59.3	60.3	61.1	62.0	62.9	63.7	64.5	65.3	66.1	66.9	67.7
0.350	61.7	62.6	63.5	64.5	65.4	66.3	67.1	68.0	68.8	69.6	70.5	71.2
0.400	64.9	65.8	66.8	67.8	68.7	69.6	70.5	71.4	72.2	73.1	73.9	74.7
0.450	68.0	69.0	70.0	71.0	71.9	72.9	73.8	74.7	75.6	76.4	77.3	78.1
0.500	71.2	72.2	73.2	74.2	75.1	76.1	77.0	78.0	78.8	79.7	70.5	81.4
0.550	74.3	75.3	76.3	77.3	78.3	79.2	80.2	81.1	82.0	82.9	83.7	84.6
0.600	77.4	78.4	79.4	80.4	81.4	82.3	83.3	84.2	85.1	86.0	86.9	87.7
0.650	80.4	81.4	82.4	83.4	84.4	85.4	86.3	87.3	88.2	89.1	90.0	90.8
0.700	80.3	84.4	85.4	86.4	87.4	88.4	89.3	90.3	91.2	92.1	93.0	93.8

Appendix IV. *(Continued)*

Group B $\sinh^{-1}((x+c)/b)^{1/2}$ transformation

c \ b	0.36	0.41	0.46	0.51	0.56	0.61	0.65	0.71	0.76	0.81	0.86	0.91
0.000	12.2	12.2	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.2
0.050	12.4	12.4	12.5	12.6	12.7	12.8	12.8	12.9	13.0	13.1	13.2	13.2
0.100	12.8	12.9	13.0	13.2	13.3	13.4	12.5	13.6	13.7	13.8	13.9	14.0
0.150	13.3	13.5	13.6	13.8	13.9	14.0	14.2	14.3	14.4	14.6	14.7	14.8
0.200	13.9	14.0	14.2	14.4	14.5	14.7	14.8	15.0	15.1	15.3	15.4	15.6
0.250	14.4	14.6	14.8	15.0	15.2	15.3	15.5	15.7	15.9	16.0	16.1	16.3
0.300	15.0	15.2	15.4	15.6	15.8	16.0	16.1	16.3	16.5	16.7	16.8	17.0
0.350	15.6	15.9	16.0	16.3	16.5	16.6	16.8	17.0	17.2	17.3	17.5	17.7
0.400	16.3	16.5	16.7	16.9	17.0	17.3	17.5	17.7	17.8	18.0	18.2	18.4
0.450	16.9	17.1	17.3	17.5	17.7	17.9	18.1	18.3	18.5	18.7	18.8	19.0
0.500	17.5	17.7	17.9	18.1	18.3	18.6	18.5	18.9	19.1	19.3	19.5	19.7
0.550	18.1	18.3	18.5	18.8	19.0	19.2	19.4	19.6	19.8	19.9	20.1	20.3
0.600	18.7	18.9	19.1	19.4	19.6	19.8	20.0	20.1	20.4	20.6	20.7	20.9
0.650	19.3	19.5	19.7	20.0	20.2	20.4	20.6	20.8	21.0	21.2	21.4	21.5
0.700	19.9	20.1	20.3	20.6	20.8	21.0	21.2	21.4	21.6	21.8	21.9	22.1

Appendix IV. (*Continued*)

Group C $\sinh^{-1}((x+c)/b)^{1/2}$ transformation

c \ b	50.45	50.50	50.55	50.60	50.65	50.70	50.75	50.80	50.85	50.90	50.95	51.00
0.000	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7
0.050	30.8	30.8	30.8	30.8	30.8	30.8	30.8	30.8	30.8	30.8	30.9	30.9
0.100	32.1	32.1	32.1	32.1	32.1	32.1	32.1	32.2	32.2	32.2	32.2	32.2
0.150	33.4	33.4	33.4	33.4	33.4	33.4	33.4	33.4	33.4	33.4	33.4	33.4
0.200	34.6	34.6	34.6	34.6	34.6	34.6	34.6	34.6	34.6	34.6	34.7	34.7
02.50	35.8	35.8	35.8	35.8	35.8	35.8	35.8	35.8	35.8	35.8	35.8	35.8
0.300	36.9	36.9	36.9	36.9	36.9	36.9	37.0	37.0	37.0	37.0	37.0	37.0
0.350	38.0	38.0	38.0	38.0	38.0	38.0	38.0	38.0	38.1	38.1	38.1	38.1
0.400	39.1	39.1	39.1	39.1	39.1	39.1	39.1	39.1	39.1	39.1	39.2	39.1
0.450	40.1	40.1	40.1	40.1	40.1	40.2	40.2	40.2	40.2	40.2	40.2	40.2
0.500	41.1	41.1	41.1	41.2	41.2	41.2	41.2	41.2	41.2	41.2	41.2	41.2
0.550	42.1	42.1	42.1	42.1	42.2	42.2	42.2	42.2	42.2	42.2	42.2	42.2
0.600	43.1	43.1	43.1	43.1	43.1	43.1	43.1	43.1	43.1	43.1	43.2	43.2
0.650	44.0	44.0	44.0	44.1	44.1	44.1	44.1	44.1	44.1	44.1	44.1	44.1
0.700	45.0	45.0	45.0	45.0	45.0	45.0	45.0	45.0	45.0	45.0	45.0	45.0

Appendix IV. (*Continued*)

Arcsinh $(cx)^{1/2}$ transformation

Group A

c:	0.78	0.83	0.88	0.93	0.98	1.03	1.08	1.13	1.18	1.23	1.28	1.33	1.38
M	41.6	41.4	41.3	41.2	41.1	41.0	41.0	41.1	41.1	41.1	41.2	41.3	41.4

Group B

c:	1.33	1.38	1.43	1.48	1.53	1.58	1.63	1.68	1.73	1.78	1.83	1.88	1.93
M	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1

Group C

c:	0.00*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
M	32.6	34.0	46.7	60.7	74.5	87.7	100.1	111.9	123.1	133.7	143.8	153.4	162.6

ln $(x+c)$ transformation

Group A

c:	0.00*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
M	130.5	49.7	43.4	41.6	41.8	43.0	44.8	47.0	49.5	52.2	55.0	57.9	60.8

Group B

c:	0.00*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
M	13.9	12.4	12.2	12.3	12.6	13.0	13.4	13.9	14.5	15.0	15.6	16.2	16.8

Group C

c:	2.44	2.49	2.54	2.59	2.64	2.69	2.74	2.79	2.84	2.89	2.94	2.99	3.04
M	38.8	38.7	38.6	38.5	38.4	38.4	38.4	38.4	38.4	38.5	38.5	38.6	38.7